FP.1

Further
Pure Maths I

Polar Coordinates
Exercise.

Suresh Goel
(Director)
Alliance World School,
Noida, Delhi-NCR
INDIA
1. The curve $C$ has polar equation:
   \[ r = 2 + 2 \cos \theta \quad 0 \leq \theta \leq \pi \]
   (a) Sketch $C$. 
   (b) Find the area of the region enclosed by $C$ and the initial line.
   (c) Show that the Cartesian equation of $C$ can be expressed as:
   \[ 4(x^2 + y^2) = (x^2 + y^2 - 2x)^2 \]

2. The curve $C_1$ has polar equation $r^2 = 2b$ for $0 \leq \theta \leq \frac{1}{2} \pi$
   (i) The point on $C_1$ furthest from the line $\theta = \frac{1}{2} \pi$
   is denoted by $P$. Show that at $P$,
   \[ 2b \tan \theta = 1 \]
   and verify that the equation has a root between 0.6 and 0.7.
   (ii) The curve $C_2$ has polar equation $r^2 = 2b \sec \theta$
   for $0 \leq \theta \leq \frac{1}{2} \pi$. The curves $C_1$ and $C_2$ intersect at the pole, denoted by $O$, and at another point $Q$.
   (iii) Find the exact value of $\theta$ at $Q$.
   (iv) Sketch $C_1$ on the diagram.

(iv) Find, in exact form, the area of the region $OQA$ enclosed by $C_1$ and $C_2$. 

[5-19 11 Q11]
2. The curve $C$ has polar equation,

$$r^2 = \ln (1 + \theta), \quad \text{for } 0 < \theta \leq 2\pi$$

(i) Sketch $C$ \hspace{1cm} \[2 \text{ marks}\]

(ii) Using substitution $u = 1 + \theta$ or otherwise, find the area of the region bounded by $C$ and the initial line, leaving your answer in exact form. \[5 \text{ marks}\]

4. The curves $C_1$ and $C_2$ have polar equations,

$$C_1: r = e^{\theta} + e^{-\theta}$$
$$C_2: r = e^{2\theta} - e^{-2\theta}$$

The curves intersect at point $P$ where $\theta = \pi$.

(i) Show that $e^{3\theta} - 2e^{\theta} - 1 = 0$. Hence find the exact value of $\theta$ and show that the value of $r$ at $P$ is $1/\sqrt{3}$. \[6 \text{ marks}\]

(ii) Sketch $C_1$ and $C_2$ on the same diagram. \[3 \text{ marks}\]

(iii) Find the area of the region enclosed by $C_1$, $C_2$, and the initial line, giving your answer correct to 3 significant figures. \[10 \text{ marks}\]

5. The curve has polar equation,

$$r = 6\cos 2\theta, \quad \text{for } -\pi/4 < \theta < \pi/4$$

(i) Sketch $C$. \[2 \text{ marks}\]

(ii) Find the area of the region enclosed by $C$. Showing full working, \[13 \text{ marks}\]

(iii) Find a cartesian equation of $C$. \[3 \text{ marks}\]

6. The curves $C_1$ and $C_2$ have polar equation as follows.

$$C_1: r = a, \quad 0 < \theta < \pi$$
$$C_2: r = 2a \cos \theta, \quad \theta \\
$$

where $a$ is a positive constant. The curves intersect at the points $P$ and $Q$. (continued)
Polar Coordinates

6. (i) Find the polar coordinates of $P_1$ and $P_2$. 
   (ii) In a single diagram, sketch $C_1$, $C_2$, and their line of symmetry. 
   (iii) The region $R$ enclosed by $C_1$ and $C_2$ is bounded by the arcs $O_1P_1$, $P_1P_2$, and $P_2O$, where $O$ is the pole. Find the area of $R$, giving your answer correct to 3 decimal places. 

7. The curve $C$ has polar equation $r = 5\sqrt{\cos 2\theta}$, where $0 < \theta < \frac{1}{2}\pi$. 
   (i) Find the area of the finite region bounded by $C$ and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. 
   (ii) Find the distance of $P$ from the initial line, giving your answer correct to 1 decimal place. 
   (iii) Find the maximum distance of $C$ from the initial line. 
   (iv) Sketch $C$. 

8. The curve $C$ has polar equation $r = a \cos 3\theta$, for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, where $a$ is a positive constant. 
   (i) Sketch $C$. 
   (ii) Find the area of the region enclosed by $C$, showing full working. 
   (iii) Using the identity $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$, find a cartesian equation of $C$. 

9. The curve $C$ has polar equation $r = a(1 + \sin \theta)$, for $-\pi \leq \theta \leq \pi$, where $a$ is a positive constant. 
   (i) Sketch $C$. 
   (ii) Find the area of the region enclosed by $C$. 

Scanned with CamScanner
10. A curve C has polar equation $r = 2a \cos \left( \frac{3\theta}{2} \right)$, for $0 \leq \theta \leq 2\pi$, where $a$ is a positive constant.
   (i) Show that $r = -2a \sin 2\theta$ and sketch C. $-14$
   (ii) Deduce that the cartesian equation of C is,
   $(x^2 + y^2)^{3/2} = -4a^2 x y$ $-2$
   (iii) Find the area of one loop of C. $-5$
   (iv) Show that, at the points other than the pole, at which a tangent to C is parallel to the initial line, $2\tan \theta = -\tan 2\theta$ $[5-17/13/87]$ $-13$

11. The polar equation of C is $r = a \left( 1 + 6 \cos \theta \right)$ for $0 \leq \theta \leq 2\pi$, where $a$ is a positive constant.
   (i) Sketch C. $-12$
   (ii) Show that the cartesian equation of C is,
   $x^2 + y^2 = a \left( 2 + \sqrt{2} x^2 + y^2 \right)$ $-6$
   (iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{\pi}{2}$. $-14$
   (iv) Find the arc length of C between the points where $\theta = 0$ and the point where $\theta = \frac{\pi}{2}$ $[5-17/11/23]$ $-11$

12. A curve has polar equation $r^2 = 8 \cos 2\theta$ for $0 < \theta < \frac{\pi}{2}$.
   Find a cartesian equation of C. $-13$
   Sketch C. $-6$
   Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, the half-line $\theta = \frac{\pi}{2}$ and the half-line $\theta = \frac{3\pi}{2}$ $-33$
   $[It is given that \theta = \ln \left( \tan \frac{\pi}{4} \right) + c]$ $[5-16/11/14]$

13. The curve C has polar equation $r = e^{4\theta}$ for $0 \leq \theta \leq \alpha$, where $\alpha$ is measured in radians. The length of C is 2015. Find the value of $\alpha$. $-6$
   $[5-17/13/23]$
14. The curves $C_1$ and $C_2$ have polar equations:

$C_1: \rho = \frac{1}{2}, \quad \text{for } 0 \leq \theta < 2\pi$

$C_2: \rho = \sqrt{2} \sin \frac{\theta}{2}, \quad \text{for } 0 \leq \theta < 2\pi$

Find the polar coordinates of the point of intersection of $C_1$ and $C_2$. \text{[2]}$

Sketch $C_1$ and $C_2$ on the same diagram. \text{[3]}

Find the exact value of the area of the region enclosed by $C_1$, $C_2$ and the half line $\theta = 0$. \text{[4]}$

5. The curve $C$ has polar equation $\rho = a \left(1 - \cos \theta \right)$

for $0 \leq \theta \leq 2\pi$

(i) Sketch $C$. \text{[2]}

(ii) Find the area of the region enclosed by arc $C$ for which $\pi \leq \theta \leq 3\pi$, the half line $\theta = \pi$ and the half line $\theta = \frac{3\pi}{2}$. \text{[5]}

(iii) Show that:

$$\left( \frac{d\rho}{d\theta} \right)^2 = 4a^2 \sin^2 \left( \frac{\theta}{2} \right)$$

where $s$ denotes the length, and find the length of the arc $C$ for which $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. \text{[7]}

W-15 11 Q11
Polar Coordinates

1. \( r = 2 + 2 \cos \theta, \ 0 \leq \theta \leq \pi \)

\[ \begin{array}{c|c|c|c|c|c|c|c} \hline \theta & 0 & \frac{\pi}{4} & \frac{\pi}{4} & \frac{\pi}{2} & \frac{\pi}{2} & \frac{3\pi}{2} & \pi \\hline r & 2 & 3 & \frac{11}{4} & 1 & \frac{11}{4} & 3 & 2 \\hline \end{array} \]

Answer continued

2(i) Hence the eqn \( 2b \cos \theta = 1 \) has a root between 0 and \( \pi \) as change of sign.

(ii) \( c_1: r^2 = 2a \)
\( c_2: r^2 = b \cos^2 \theta \)

for point of intersection form \( 2b \) \( \Rightarrow \) \( a \cos^2 \theta = 2a \cos \theta \)
\( \Rightarrow a \cos \theta = \sqrt{2} \)
\( \theta = \frac{\pi}{4} \)

(b) Area \( = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (2 + 2 \cos \theta)^2 d\theta \)
\( = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (4 + 8 \cos \theta + 4 \cos^2 \theta) d\theta \)
\( = \frac{1}{2} \left[ \left( 3 + 4 \cos \theta + 4 \cos^2 \theta \right) \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} \)
\( = \left[ \frac{3 \pi}{2} + 4 \sin \theta + \frac{4}{6} \sin 2\theta \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} \)
\( = \frac{3 \pi}{2} \)

(c) \( \lambda = 2 + 2 \cos \theta \)
\( x = 2 + \cos \theta \)
\( y = \sin \theta \)
\( \sqrt{x^2 + y^2} = \sqrt{(2 + \cos \theta)^2 + \sin^2 \theta} \)
\( 2 \sqrt{x^2 + y^2} = 2 + 2 \cos \theta \)
\( \Rightarrow 4 (x^2 + y^2) = (2 + 2 \cos \theta)^2 \)

2. Given \( x^2 = 2a \)
\( 0 \leq \theta \leq \pi \)

(i) \( \lambda = \sqrt{2} \theta \)
\( x = 2 \sqrt{2} \theta \cos \theta \)
\( \Rightarrow \frac{dx}{d\theta} = \sqrt{2} \theta \cos \theta - 2 \sqrt{2} \sin \theta \)

For 0 \( \theta = 0 \Rightarrow \theta = 0, 2 (1.6), 6 (0.6) \) \( -1 = -0.179 \)
\( \theta = 0.17 \Rightarrow \theta = 0.17, 6 (0.7) \) \( -1 = 0.179 \)

Area \( \text{OPQ} = \int_{0}^{\pi} \frac{1}{2} 2b \cos \theta - \int_{0}^{\pi} b \cos \theta \cos \theta d\theta \)
\( = \frac{1}{2} \int_{0}^{\pi} \theta (2 - \cos^2 \theta) d\theta \)
\( = \frac{1}{2} \left[ \frac{\pi}{2} (2 - \cos^2 \theta) \right]_{0}^{\pi} \)
\( = \frac{1}{2} \left[ (2 \pi) - \frac{\pi}{2} \right] \)
\( = \frac{3}{2} \pi \)

\( \theta = \frac{\pi}{2} \)
3. (i) Given curve \( C: x^2 = \ln(1 + \theta) \), \( 0 \leq \theta \leq 2\pi \\
4(\text{iii}) A = \frac{1}{2} \int_0^\pi \left[ 4(e^\theta + e^{-\theta})^2 \right] d\theta \\
\begin{align*}
&= \frac{1}{2} \int_0^\pi \left[ 16(e^{2\theta} + 2 + e^{-2\theta}) \right] d\theta \\
&= \frac{1}{2} \int_0^\pi \left[ 16(e^{2\theta} + 2 + e^{-2\theta}) \right] d\theta \\
&= \frac{1}{2} \int_0^\pi \left[ 5 + 2e^{2\theta} + 2 - 2\theta \right] \left[ \theta - e^{-2\theta} \right] d\theta \\
&= \frac{1}{2} \left[ 5\theta + e^{2\theta} - e^{-2\theta} + e^\theta + e^{-\theta} \right]_0^\pi \\
&= 5\ln(1 + \sqrt{2}) + (1 + \sqrt{2})^2 - (1 + \sqrt{2})^{-2} \\
&= \frac{5\pi}{8} \\
\end{align*}

(ii) \( \int_0^{2\pi} x^2 \, dx = \frac{1}{2} \int_0^{2\pi} \ln(1 + \theta) \, d\theta \\
= \frac{1}{2} \left[ \theta \ln(1 + \theta) - \theta \right]_0^{2\pi} \\
= \frac{1}{2} \left[ 2\pi \ln(1 + 2\pi) - 2\pi \right] \\
= \pi \ln(1 + \sqrt{2}) - \pi \\
\]

(iii) \( \int_0^{2\pi} x^2 \, dx = \pi \ln(1 + \sqrt{2}) - \pi \\
\]

4. (i) \( \theta = \cos \theta, -\pi \leq \theta \leq \pi \\
\begin{align*}
&\begin{array}{cccc}
\theta & -\frac{\pi}{4} & -\frac{\pi}{4} & 0 & \frac{\pi}{4} \\
\lambda & 0 & 1 & 0 & \frac{\pi}{4}
\end{array} \\
&\begin{array}{cccc}
\lambda & 0 & 1 & 0 & \frac{\pi}{4}
\end{array} \\
\end{align*}

(ii) \( A = \frac{1}{2} \int_0^{2\pi} \cos^2 \theta \, d\theta \\
= \frac{1}{4} \int_0^{2\pi} (\cos 4\theta + 1) \, d\theta \\
= \frac{1}{4} \left[ \theta + \sin 4\theta \right]_0^{2\pi} \\
= \frac{\pi}{8} = 0.393 \\
\]

(iii) \( \lambda = \cos \theta, \quad \lambda = \frac{\cos^2 \theta - \sin^2 \theta}{\lambda^2 - \frac{y^2}{2}} \\
\lambda^2 = \frac{x^2 + y^2}{2} \\
\lambda^3 = x^2 - y^2 \\
\]

\[\text{Scan with CamScanner}\]
\[ a = 2a/3 \Rightarrow \cos \theta = \pm \frac{1}{2} \]
\[ \theta = \left( \frac{\pi}{3}, \frac{2\pi}{3} \right) \]

\[ A = \frac{1}{2} \int_{a}^{2a} x \, dx = \frac{3}{2} \int_{0}^{\pi/3} 2a \cos \theta \, d\theta \]
\[ = \frac{3}{2} \left[ 2a \sin \frac{\pi}{3} \right]_{0}^{\pi/3} = 2a^2 \sqrt{3} \]
\[ = -\frac{3}{2} \ln \left( \frac{1}{2} \right) = 57.6 \checkmark \]

\[ (iii) \frac{dy}{d\theta} = \frac{5}{12} \cdot \frac{1}{\sqrt{4 - 2x}} \times 2 \cos 2\theta x = 0 \]

or \[ \max (\sin 2\theta) = 1 \]
\[ y = \frac{a^2}{2} \sin 2\theta = 3.54 \checkmark \]

\[ \theta = \pi/3, -\pi/3 \leq \theta \leq \pi/3 \]

and, \( a \) is constant

\[ A = \frac{1}{2} \int_{a}^{2a} \frac{a^2}{6} \cos^2 \theta \, d\theta \]
\[ = \frac{a^2}{4} \int_{0}^{\pi/3} \left( \frac{1}{2} + \frac{3}{2} \cos 2\theta \right) \, d\theta \]
\[ = \frac{a^2}{4} \left[ \frac{\theta}{2} + \frac{3}{4} \sin 2\theta \right]_{0}^{\pi/3} = \frac{\pi a^2}{12} \]

(Continued →)
8(iii) \( r = a \cos 3\theta \)

\[
= a \left( 4 \cos^3 \theta - 3 \cos \theta \right)
\]

\[
= a \cos \theta \left( 4 \cos^2 \theta - 3 \right)
\]

\[
\Rightarrow r = a \left( \frac{x}{r} \right) \left[ 4 \left( \frac{x}{r} \right)^2 - 3 \right]
\]

\[
\Rightarrow r^2 = a \left( 4x^2 - 3x^2 + 3 \right)
\]

\[
\Rightarrow (x^2 + y^2)^2 = a \left( 4x^2 - 3 \right)(x^2 + y^2)
\]

\[
\Rightarrow (x^2 + y^2)^2 = a \left( 4x^2 - 3x^2 + y^2 \right)
\]

\[
\Rightarrow (x^2 + y^2)^2 = a \left( 4x^2 - 3x^2 + y^2 \right)
\]

9. \( C : \theta = a(1 + \sin \theta) \), \(-\pi < \theta < \pi \)

\( a > 0 \) constant

(i) \[
\theta = a(1 + \sin \theta)
\]

(ii) \[
A = \frac{1}{2} \int_{-\pi}^{\pi} (1 + 2 \sin \theta + \sin^2 \theta) \, d\theta
\]

\[
= \frac{1}{2} \int_{-\pi}^{\pi} \left( 1 + 2 \sin \theta + \frac{1}{2} \left( 1 - \cos 2\theta \right) \right) \, d\theta
\]

\[
= \frac{1}{2} \left[ 2\pi - 2 \sin \theta - \frac{1}{2} \sin 2\theta \right]_{-\pi}^{\pi}
\]

\[
= \frac{3}{2} \pi a^2
\]

10. \( C : r = 2a(\cos (\theta + \frac{\pi}{2})) \), \( 0 < \theta < \frac{\pi}{2} \)

\( a > 0 \) constant

(i) \[
r = 2a \left[ \cos \left( \theta + \frac{\pi}{2} \right) \right]
\]

\[
r = -2a \sin \theta \quad \checkmark
\]

(ii) \[
r = -4a \sin \theta \quad \checkmark
\]

\[
r^3 = -4ax
\]

\[
(x^2 + y^2)^{\frac{3}{2}} = -4ax
\]

10(iii) Area one leaf.

\[
= \frac{1}{2} \int_{\pi}^{\frac{\pi}{2}} 4x^2 \sin^2 2\theta \, d\theta
\]

\[
= \frac{1}{2} \int_{\pi}^{\frac{\pi}{2}} (1 - \cos 4\theta) \, d\theta
\]

\[
= \frac{1}{2} \left[ \theta - \frac{1}{4} \sin 4\theta \right]_{\pi}^{\frac{\pi}{2}}
\]

\[
= \frac{\pi}{2} a^2 \quad \checkmark
\]

10(iv) \( y = 2 \sin \theta \), \( \frac{dy}{d\theta} = 2 \cos \theta \)

\[
dy = 2a \left[ 2 \cos 2\theta + \sin 2\theta \right] = 0
\]

\[
\Rightarrow \cos 2\theta + \sin 2\theta = 0
\]

\[
\Rightarrow 2 \cos 2\theta = -\sin 2\theta \quad \checkmark
\]

11. \( C : r = a(1 + \cos \theta) \) for \( 0 < \theta < 2\pi \)

\( a > 0 \) constant

(i) \[
r = a(1 + \cos \theta)
\]

(ii) \[
A = \frac{1}{2} \int_{0}^{\frac{\pi}{2}} (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \left( \frac{3}{2} + 2 \cos 2\theta + 2 \cos^2 \theta \right) \, d\theta
\]

\[
= \frac{1}{2} \left[ \frac{3}{2} + 2 \sin 2\theta + 2 \left( \frac{1}{2} + \frac{1}{2} \sin 2\theta \right) \right]_{0}^{\frac{\pi}{2}}
\]

\[
= \frac{9}{16} \pi + 9 \sqrt{3} \quad \checkmark
\]

10. \( \text{arc length} = xx \ldots \)

Scanned with CamScanner
Polar Coordinates

12. \[ r^2 = 8 \cos 2\theta \quad \text{for} \quad 0 \leq \theta \leq \pi \]

\[ \Rightarrow r^2 = \frac{8}{\cos 2\theta} \]

\[ \Rightarrow r^2 = \frac{8}{2 \cos^2 \theta - 1} \]

\[ = 4 \csc^2 \theta \]

\[ \Rightarrow 2 \csc \theta = 4 \Rightarrow \csc \theta = 2 \]

\[ \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \]

Area \( A = \int_{\pi/6}^{5\pi/6} \frac{8}{\cos 2\theta} \, d\theta \)

\[ = \frac{8}{2} \left[ \ln |\csc \theta - \cot \theta| \right]_{\pi/6}^{5\pi/6} \]

\[ = -2 \left[ \ln |\csc \frac{5\pi}{6} - \cot \frac{5\pi}{6}| - \ln |\csc \frac{\pi}{6} - \cot \frac{\pi}{6}| \right] \]

\[ = 4 \ln 3 \]

13. \[ x = e^\theta \quad \text{for} \quad 0 \leq \theta \leq \pi \]

\[ s = \int_0^\pi \sqrt{e^{2\theta} + 16 e^\theta} \, d\theta \]

\[ = \int_0^\pi e^\theta \sqrt{4 + e^{2\theta}} \, d\theta \]

\[ = \frac{1}{4} \left[ 4e^\theta - \frac{e^{2\theta}}{2} \right]_0^\pi \]

\[ = 20.15 \]

\[ \Rightarrow e^\alpha = 19.55 \]

\[ \Rightarrow \alpha = 1.89 \]

14. \[ c_1: \quad r = \frac{1}{2} \quad \text{for} \quad 0 \leq \theta < 2\pi \]

\[ c_2: \quad r = \sqrt{8 \sin 2\theta} \quad \text{for} \quad 0 \leq \theta < 2\pi \]

For point of intersection:

\[ \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \]

Point \( (\frac{\pi}{6}, \frac{\pi}{3}) \)

15. \[ c: \quad x = a \left( 1 - \cos \theta \right) \quad 0 \leq \theta \leq 2\pi \]

\[ A = \frac{1}{2} \int_0^{2\pi} (1 - 2a \cos \theta + a^2 \sin^2 \theta) \, d\theta \]

\[ = a^2 \int_0^{2\pi} \left( \frac{3}{2} - 2a \cos \theta + \frac{1}{4} \sin^2 \theta \right) \, d\theta \]

\[ = a^2 \left( \frac{3\pi}{2} - 2a \sin \theta + \frac{1}{4} \sin^2 \theta \right)_0^{2\pi} \]

\[ = a^2 \left( \frac{3\pi}{2} + 2 \right) \]