Further
Pure Maths 1

Proof by Induction
Exercise.

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1. It is given that, \(\phi(n) = 5^n (4n+1) - 1\), for \(n = 1, 2, 3, \ldots\)
Prove, by mathematical induction that \(\phi(n)\) is divisible by 8, for every positive integer \(n\). - [7]

2. (i) Prove by mathematical induction that, for \(x \neq 1\) and all positive integers \(n\),
\[
1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}
\]
- [5]

3. Prove by mathematical induction that,
\[3^{3n} - 1\] is divisible by 13 for every positive integer \(n\). - [5]

4. It is given that \(f(n) = 3^n + 8^{n-1}\). By simplifying \(f(k) + f(k+1)\), or otherwise, prove by mathematical induction that \(f(n)\) is divisible by 9, for every positive integer \(n\). - [6]

5. For the sequence \(u_1, u_2, u_3, \ldots\), it is given that \(u_1 = 8\) and \(u_{n+1} = 5u_n - 3\) for all \(n\).
(i) Prove by mathematical induction that,
\[u_n = 4 \left(\frac{5}{4}\right)^n + 3\]
for all positive integer \(n\). - [5]
(ii) Deduce the set of values of \(x\), for which the infinite series, \((u_1-3)x + (u_2-3)x^2 + \ldots + (u_n-3)x^n + \ldots\) is convergent. - [2]
(iii) Use the result given in part (i) to find bounds \(a\) and \(b\) such that,
\[
\sum_{n=1}^{\infty} \ln (u_n-3) = N^2 \ln a + N \ln b.
\]
- [3]
6. The sequence of positive numbers $U_1, U_2, U_3, \ldots$ is such that $U_1 < 3$ and for $n > 1$,

$$U_{n+1} = \frac{4U_n + 9}{U_n + 4}$$

(i) By considering $3-U_{n+1}$, or otherwise, prove by mathematical induction that $U_n < 3$ for all positive integers $n$. -- [5]

(ii) Show that $U_{n+1} > U_n$ for $n > 1$. -- [3]

7. Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers $n$. -- [5]

8. Prove, by mathematical induction, that

$$\sum_{i=1}^{n} \frac{\ln \left( \frac{n+1}{i} \right)}{i} = n \ln \left( \frac{n+1}{n} \right)$$

for all positive integers $n$. -- [6]

9. Prove by mathematical induction that, for all positive integers $n$, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. -- [6]

10. It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that the $n$-sided polygon has $\frac{1}{2} n (n-3)$ diagonals, when $n > 3$. -- [6]

11. Using factorials show that, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ -- [2]

Hence prove by mathematical induction that:

$$(a+x)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} x + \cdots + \binom{n}{n} x^n$$

for all positive integer $n$. -- [4]
12. The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n + 5}{5}$ for every positive integer $n$. Prove by mathematical induction that $a_n > 5$ for every positive integer $n$. Prove also that $a_n > a_{n+1}$ for every positive integer $n$. 

13. Prove by mathematical induction that, for all positive integer $n$, $\sum_{i=1}^{n} \frac{1}{(2i)^2 - 1} = \frac{n}{2n+1}$. 

State the value of $\sum_{i=1}^{\infty} \frac{1}{(2i)^2 - 1}$.
Proof by Induction

\[ \phi(m) = 5^m(4^m+1) - 1 \]

1. \[ \phi(1) = 5 \times 5 - 1 = 24 \text{ which is divisible by } 8 \]
   
   Assume \( \phi(k) \) is true for some positive integer \( k \).

   Consider \( \phi(k+1) - \phi(k) = 8l \)
   
   \[ \phi(k+1) = 8l + \phi(k) = 8l + 8m \]
   
   \( \phi(k+1) = 8(l+m) \)

   \( \therefore \phi(n) \) is true for all \( n \), using PMI.

Alternate method:

\( \phi(1) = 5 \times 5 - 1 = 24 \text{ which is divisible by } 8 \)

Now let \( \phi(1) \) be true for some positive integer \( k \).

Now \( \phi(k+1) = 5^{k+1}(4^{k+1}+1) - 1 \)

\[ D = 5(8l - 5^k + 1) + 25 \times 5^k - 1 \]

\[ D = 40l + 25^k + 4 \]

\[ = 40l + 4(5^k + 1) \]

\[ = 40l + 4(5^k + 1) \]

\[ = 8(5l + m)(8k^3 - 1) \]

So \( \phi(k+1) \) is divisible by 8.

\( \therefore \phi(n) \) is true for all positive integers, using PMI.

2. \[ \text{for } n = 1, \ 1 = \frac{1}{2} \text{ is True} \]
   
   \( \therefore \phi(1) \) is True.

3. \[ \text{for } n = 1, \ 3^2 - 1 = 26 \text{ is divisible by } 13 \]
   
   Assume \( \phi(k) \) is true for all positive integers.

4. \[ \text{for } n = 1, \ 3^3 - 1 = 26 \text{ is divisible by } 13 \]
   
   \[ (3^{3k} - 1) \text{ is divisible by } 13 \]

5. \[ \text{for } n = k + 1, \ 3^{3(k+1)} - 1 = 27 \times 3^{3k} - 1 \]

6. \[ \text{The first term is divisible by } 13 \text{ from (i)} \]

7. \[ \text{and second term clearly divides any PMI}\]

8. \[ \text{the statement is true for all positive integers} \]

9. \[ \phi(1) = 9 \text{ divisible by } 9 \]
   
   \( \phi(k) \) is true.

10. Assume \( \phi(k) \) is true.

11. \[ \text{for } (2^{3k} + 8^{k-1}) \text{ is divisible by } 9 \]

12. \[ \text{Now consider } \phi(k+1) + \phi(k) \]

13. \[ = 2^{3k} + 8^{k-1} + 2^{3k} + 8^{k-1} \]

14. \[ = 9 \times 2^{3k} + 8^{k-1} \]

15. \[ \phi(n) \] is true for all positive integers.
5. (i) \( P_n : u_n = 4 \left( \frac{5}{4} \right)^n + 3 \)

\[\text{let } n = 1, \Rightarrow 4 \left( \frac{5}{4} \right)^1 + 3 = 8 \Rightarrow P_1 \text{ is true.} \]

\[\text{let } P_k \text{ be true for some } k, \text{ then } \]

\[u_{k+1} = 4 \left( \frac{5}{4} \right)^{k+1} + 3 = 4 \left( \frac{5}{4} \right)^{k} \times 5 + 3 = 4 \left( \frac{5}{4} \right)^k \times 5 + 3 \]

\[\Rightarrow P_k \Rightarrow P_{k+1}, \Rightarrow u_n < 3 \text{ for all positive integers.} \]

\[\text{(iii) } (u_n-3) = 4x^n \left( \frac{5}{4} \right)^n \]

\[\text{so the series is convergent for } -1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5} \]

\[\frac{N}{n} \sum_{n=1}^{N} ln \left( \frac{u_n-3}{u_n} \right) = \sum_{n=1}^{N} ln \left( \frac{4 \left( \frac{5}{4} \right)^n}{4} \right) = \sum_{n=1}^{N} \frac{5}{4} \ln n - \frac{4}{4} \ln n + \frac{5}{4} \ln n = 5 \ln \left( \frac{N+1}{N} \right) + 5 \ln \left( \frac{2\sqrt{5}}{2} \right) \]

\[\Rightarrow a = \sqrt{5}, \ b = 2\sqrt{5} \]

6. (i) \( u_1 < 3 \text{ (given)} \]

Assume that: \( u_k < 3 \)

Then \( 3 - u_{k+1} = 3 - 4(u_k + 3) = -u_k + 3 \)

\[\Rightarrow u_{k+1} < 3 \]

\[\text{so by PMI, } u_n < 3 \text{ for all } n \geq 1 \]

\[\text{(ii) } u_{n+1} - u_n = 4u_n + 9 - u_n = 3u_n + 9 \]

\[\Rightarrow u_{n+1} - u_n \geq 0 \]

So \( u_n < 3 \Rightarrow u_{n+1} - u_n \geq 0 \)
9. For $n=1$, $10+192+5$  
$= 207 = 9 \times 23 \Rightarrow P$ is true.

10. Let $P_k$ be true for some positive integer $k$.

Then $P_{k+1}$ is also true.

$c) P_{k+1} - P_k = 10^k(10-1) + 3 \times \left(4^k + 2\right) \left(4^k + 1\right)$

$= 9 \left(10^k + 4^k + 2\right)$

$= 9 \beta$.

$P_{k+1} - 9 \alpha = 9 \beta$  \(\times\)

$P_{k+1} = 9(\beta + \alpha)$


11. $(\begin{array}{c} n \\ k \end{array}) + (\begin{array}{c} n \\ k+1 \end{array})$  

$= \frac{n!}{(n-k)! (n-k+1)!} + \frac{n!}{(n-k)!(n-k+1)!}$

$= \frac{n!}{(n-k)!(n-k+1)!} \left[ \frac{1}{x} + \frac{1}{x} \right]$  

$= \frac{n!}{(n-k+1)! x} \left[ \frac{1}{x} \right]$

$= \frac{n!}{(n-k+1)! x} \left[ \frac{1}{x} \right]$

Now $(a+x)' = (\begin{array}{c} 1 \\ 0 \end{array}) a + (\begin{array}{c} 1 \\ 1 \end{array}) x = a + x$

Now let $P_k$ be true.

Then $P_{k+1}$ is also true.

$a+ (a+x)^k = (\begin{array}{c} 0 \\ k \end{array}) a^k + (\begin{array}{c} k \\ 1 \end{array}) a^{k-1} x + \ldots + (\begin{array}{c} k \\ k \end{array}) a^0 x^k$

Multiply by $(a+x)$ on both sides.

The above equation becomes:

$\Rightarrow P_{k+1}$ is true.

Now let $P_k$ is true.

Then $P_{k+1}$ is also true for all positive integers $n \geq 1$.

Note: "Adding an extra vertex, a function $(K-1)$ diagonals can be drawn in a $K$-gon.

\(\Rightarrow\) $P_{k+1}$ is true when $P_k$ is true.

Using PMI, the given statement is true for all positive integers $n \geq 1$.\)
12. \( a_n > 5 \) (given) \( \Rightarrow P_1 \) is True

Let \( P_k \) is True for some positive integer \( k \), \( \Rightarrow a_k = 5 + \delta, \delta > 0 \)

Now consider \( a_{k+1} = \frac{4a_k^2 + 25 - 5}{5a_k} \)

\[ a_{k+1} = \frac{4(5 + \delta)^2 + 25 - 5}{5(5 + \delta)} = \frac{4(25 + 10\delta + \delta^2) + 20}{5(5 + \delta)} \]

\[ a_{k+1} = \frac{4(25 + 10\delta + \delta^2) + 20}{5(5 + \delta)} = \frac{a_k^2 + 8a_k + 25}{5a_k} \]

\[ a_{k+1} > 5 \]

\( \Rightarrow P_{k+1} \) is True for all positive \( n \geq 1 \).

Again \( a_{k+1} = a_k = 5 - \frac{1}{a_k} \)

As \( \delta < 1 \) and \( \frac{1}{a_k} > 1 \)

\[ \Rightarrow a_{k+1} - a_k < 0 \]

\[ \Rightarrow a_{k+1} < a_k \]

\( \Rightarrow a_{k+1} > a_n \)

13. \( P_k: \sum_{i=1}^{k} \frac{1}{(2i)^2 - 1} = \frac{k}{2k+1} \)

Let it be true for some positive integer \( k \).

Now \( \frac{1}{(2k)^2 - 1} = \frac{1}{2k+1} \)

Now \( k + 1 \)

\[ \frac{1}{(2k+2)^2 - 1} = \frac{1}{(2k+1)(2k+3)} \]

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\( \Rightarrow P_k \Rightarrow P_{k+1} \), \( \text{by } \) PMI

\( P_n \) is True for all positive \( n \geq 1 \).