1. The cubic equation \( x^3 - x^2 - x - 5 = 0 \)
   has roots \( \alpha, \beta \) and \( \gamma \).
   (a) Show that the value of \( \alpha^3 + \beta^3 + \gamma^3 \) is 19. -- [4]
   (b) Find the value of \( \alpha^4 + \beta^4 + \gamma^4 \). -- [2]
   (c) Find a cubic equation with roots \( \alpha + 1, \beta + 1 \) and \( \gamma + 1 \),
   give your answer in the form \( px^3 + qx^2 + rx + s = 0 \),
   where \( p, q, r \) and \( s \) are constants to be determined. -- [3]

2. The equation \( x^3 - x + 1 = 0 \) has roots \( \alpha, \beta \) and \( \gamma \).
   (i) Use the relation \( \chi = y^{1/3} \) to show that the equation,
       \[ y^3 + 3y^2 + 3y + 1 = 0 \]
   has roots \( \alpha^3, \beta^3, \gamma^3 \). Hence write down the value of,
   \[ \alpha^3 + \beta^3 + \gamma^3 \]. -- [3]
   \[ \text{let } S_3 = \alpha^n + \beta^n + \gamma^n \]
   (ii) Find the value of \( S_{-3} \). -- [2]
   (iii) Show that \( S_6 = 5 \) and find the value of \( S_9 \). -- [4]

3. A cubic equation \( x^3 + bx^2 + cx + d = 0 \) has real roots
   \( \alpha, \beta \) and \( \gamma \) such that, \[ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12} \]
   \[ \alpha \beta \gamma = -12 \]
   \[ \alpha^3 + \beta^3 + \gamma^3 = 90 \]
   (i) Find the values of \( c \) and \( d \). -- [3]
   (ii) Express \( \alpha^2 + \beta^2 + \gamma^2 \) in terms of \( b \). -- [2]
   (iii) Show that \( b^3 - 15b + 126 = 0 \). -- [4]
   (iv) Given that \( 3 + \sqrt{12} \) is a root of
       \[ y^3 - 15y + 126 = 0 \], deduce the value of \( b \). -- [2]
4. The equation $x^3 + 9x^2 + x - 7 = 0$ has roots $\alpha$, $\beta$, $\gamma$.
   (i) Use the relation $x^3 = -7y$ to show that the equation $42y^3 + 14y^2 - 27y + 7 = 0$
   has roots $\frac{\alpha}{\beta^2}$, $\frac{\beta}{\gamma^2}$, $\frac{\gamma}{\alpha^2}$.
   -- [4]
   (ii) Show that $\frac{\alpha^2}{\beta^2} \cdot \frac{\beta^2}{\gamma^2} + \frac{\beta^2}{\alpha^2} + \frac{\gamma^2}{\beta^2} = \frac{58}{49}$.
   -- [3]
   (iii) Find the exact value of $\frac{\alpha^3}{\beta^3} + \frac{\beta^3}{\gamma^3} + \frac{\gamma^3}{\alpha^3}$.
   -- [3]

5. It is given that the equation $x^3 - 21x^2 + kx - 216 = 0$, where $k$ is a constant, has real roots $\alpha$, $\beta$, and $\gamma$.
   (i) Find the numerical value of the roots. -- [6]
   (ii) Deduce the value of $k$. [8-18/11/94]
   -- [2]

6. The equation $9x^3 - 9x^2 + x - 2 = 0$
   has roots $\alpha$, $\beta$, $\gamma$.
   (i) Use substitution $y = 3x-1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation $y^3 - 2y - 7 = 0$.
   -- [3]
The sum $(3\alpha-1)^n + (3\beta-1)^n + (3\gamma-1)^n$ is denoted by $S_n$.
   (i) Find the value of $S_3$. -- [2]
   (iii) Find the value of $S_4$ [5-18/13/96].
   -- [4]

7. The roots of the equation $x^3 + px^2 + qx + r = 0$
   are $\alpha$, $\beta$, $\gamma$, where $p$, $q$, and $r$ are non-zero real constants.
   (i) Show that: $2\alpha + q = 0$.
   -- [4]
   (ii) Show that: $p\beta - q^2 = 0$.
   -- [2]

8. The cubic equation $8x^3 - 3x^2 + 4x - 10 = 0$ has roots $\alpha$, $\beta$, and $\gamma$.
   (i) Find the value of $(\alpha+1)(\beta+1)(\gamma+1)$. -- [4]
   (ii) Find the value of $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$. [W-18/11/94] -- [4].
9. The roots of the cubic equation,
\[ x^3 - 5x^2 + 13x - 4 = 0 \]
are \( \alpha, \beta, \gamma \).

(i) Find the value of \( \alpha^2 + \beta^2 + \gamma^2 \) \( \quad \text{[13]} \)

(ii) Find the value of \( \alpha^3 + \beta^3 + \gamma^3 \) \( \quad \text{[12]} \)

10. By finding a cubic equation whose roots are \( \alpha, \beta, \text{ and } \gamma \), solve the set of simultaneous equations.
\[ \alpha + \beta + \gamma = -1 \]
\[ \alpha^2 + \beta^2 + \gamma^2 = 29 \]
\[ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -1 \] \( \quad \text{[18]} \)

11. The roots of the cubic equation,
\[ x^3 + 2x^2 - 3 = 0 \]
are \( \alpha, \beta \) and \( \gamma \).

(i) Using the substitution \( y = \frac{1}{x^2} \), find the cubic equation with roots \( \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2} \). \( \quad \text{[13]} \)

(ii) Hence find the value of \( \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \). \( \quad \text{[11]} \)

(iii) Find also the value of \( \frac{1}{\alpha^2 \beta^2} + \frac{1}{\beta^2 \gamma^2} + \frac{1}{\gamma^2 \alpha^2} \) \( \quad \text{[11]} \)

12. The roots of a cubic equation:
\[ 2x^3 + x^2 - x = 0 \]
are \( \alpha, \beta \) and \( \gamma \). Using the substitution \( y = 1 + \frac{x}{x^2} \), find the cubic equation whose roots are \( 1 + \frac{1}{\alpha}, 1 + \frac{1}{\beta} \) and \( 1 + \frac{1}{\gamma} \), giving your answer in the form \( ay^3 + by^2 + cy + d = 0 \), where \( a, b, c \) and \( d \) are the constants to be determined. \( \quad \text{[14]} \)

13. The cubic equation \( x^3 - z^2 - x - 5 = 0 \) has roots \( \alpha, \beta \) and \( \gamma \). Show that the value of \( \alpha^3 + \beta^3 + \gamma^3 \) is 19. \( \quad \text{[14]} \)

Find the value of \( \alpha^4 + \beta^4 + \gamma^4 \) \( \quad \text{[12]} \)

Show that the cubic equation with roots \( \frac{\alpha}{\beta}, \frac{\beta}{\gamma} \) and \( \frac{\gamma}{\alpha} \) may be found using the substitution \( \text{(Continued)} \)
Polynomial Equations

13. \( x = \frac{1}{-x} \), and find this equation, giving your answer in the form \( px^3 + qx^2 + rx + s = 0 \), where \( p, q, r \) and \( s \) are constant to be determined.

14. Find the cubic equation with roots \( \alpha, \beta \), and \( \gamma \) such that:
   \[ \alpha + \beta + \gamma = 3 \]
   \[ \alpha^2 + \beta^2 + \gamma^2 = 1 \]
   \[ \alpha^3 + \beta^3 + \gamma^3 = -30 \]
   Give your answer in the form \( x^3 + px^2 + qx + r = 0 \), where \( p, q, \) and \( r \) are integer to be found.

15. The roots of cubic equation \( x^3 - px^2 + qx - 3 = 0 \) are \( \alpha, \beta \), and \( \gamma \). Find the value of:
   (i) \( \frac{1}{(\alpha \beta) \gamma \alpha} \)
   (ii) \( \frac{1}{\alpha \beta} + \frac{1}{\beta \gamma} + \frac{1}{\gamma \alpha} \)
   (iii) \( \frac{1}{\alpha^2 \beta} + \frac{1}{\alpha \beta^2} + \frac{1}{\alpha \beta \gamma} \)
   Deduce a cubic equation, with integer coefficient, having roots \( \frac{1}{\alpha \beta}, \frac{1}{\beta \gamma}, \) and \( \frac{1}{\gamma \alpha} \).

16. The quartic equation \( x^4 + px^3 + qx^2 + rx + s = 0 \), where \( p, q, r \) and \( s \) are real constants, has two pairs of equal roots, show that \( p^2 + 4q = 0 \) and state the value of \( q \).

17. The cubic equation \( x^3 + px^2 + qx + r \), where \( p, q, \) and \( r \) are integers, has roots \( \alpha, \beta, \) and \( \gamma \), such that:
   \[ \alpha + \beta + \gamma = 15 \]
   \[ \alpha^2 + \beta^2 + \gamma^2 = 83 \]
   Write down the value of \( p \) and the value of \( q \).
   Given that \( \alpha, \beta \), and \( \gamma \) are all real and that \( \alpha \beta + \gamma \alpha = 36 \), find \( \alpha \) and hence find the value of \( r \).
Polynomial Equations

1. Let $S_n$ denote $\alpha^n + \beta^n + \gamma^n$

$$S_2 = \alpha^2 + \beta^2 + \gamma^2 = \frac{1}{2} \left( \frac{1}{3} \right) = \frac{1}{6}$$

A) $S_3 = \alpha^3 + \beta^3 + \gamma^3 = -1$

$$S_3 = \alpha \cdot \beta \cdot \gamma = -1$$

$$= \frac{1}{3} \cdot 3 \cdot \left( \alpha \beta \gamma \right) = -1$$

$$= \frac{1}{3} \cdot 3 \cdot (-1) + 3 \cdot 5 = 19$$

$$\sum \alpha^3 = (-1) + \frac{1}{3} \cdot 3 \cdot (-1) + 3 \cdot 5 = 19$$

B) $S_4 = \alpha^4 + \beta^4 + \gamma^4 = \left( \sum \alpha^2 \right)^2 - 2 \left( \sum \alpha \beta \gamma \right) = \left( \sum \alpha \beta \gamma \right)^2 - 2 \left( \sum \alpha \beta \gamma \right)$

$$S_4 = \left( \sum \alpha \beta \gamma \right)^2 - 2 \left( \sum \alpha \beta \gamma \right) = 2 \left( \sum \alpha \beta \gamma \right)$$

$$= 2 \left( \sum \alpha \beta \gamma \right) = 2 \left( \sum \alpha \beta \gamma \right)$$

C) $\sum \alpha^2 = \left( \sum \alpha \right)^2 - 2 \left( \sum \alpha \beta \gamma \right)$

$$\sum \alpha^2 = \left( \sum \alpha \right)^2 - 2 \left( \sum \alpha \beta \gamma \right) = \left( \sum \alpha \right)^2 - 2 \left( \sum \alpha \beta \gamma \right)$$

$$\sum \alpha^2 = \left( \sum \alpha \right)^2 - 2 \left( \sum \alpha \beta \gamma \right) = \left( \sum \alpha \right)^2 - 2 \left( \sum \alpha \beta \gamma \right)$$

2(i) Given $x^3 + 6x^2 + 7x + 1 = 0$

$$\Rightarrow x^3 - x^2 + 6x - 1 = 0$$

Let $y = x^2$ and $\alpha = y + 5$

$$\Rightarrow y^2 - y + 5 = 0$$

3(iii) $S_3 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$$\Rightarrow \frac{\alpha^3 \beta^3 \gamma^3}{\alpha^3 + \beta^3 + \gamma^3} = \frac{2}{1}$$

continued
4(ii) \( \frac{1}{\beta^2} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} \)

\[ \sum \frac{1}{\beta} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-12}{49} = -\frac{2}{7} \]

\[ \Sigma \frac{1}{\beta} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-27}{49} \]  

\[ \Rightarrow \Sigma \frac{1}{\beta^2} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-27}{49} \]  

\[ \text{Note} \]

\[ \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\gamma^2} + \frac{\gamma^2}{\alpha^2} = \left( \frac{\alpha}{\beta} \right)^2 - 2 \left( \frac{\alpha}{\beta} \right) \]

\[ = \left( \frac{-3}{7} \right)^2 - 2 \left( \frac{-3}{7} \right) = \frac{9}{49} \]

\[ = \frac{9}{49} \]  

\[ \text{from (i)} \]

\[ \frac{1}{\beta^2} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-27}{49} \]

\[ \Rightarrow 2 \beta^2 \alpha + 9 \beta - \frac{27}{49} = 0 \]  

\[ \Rightarrow \]

\[ (\alpha^2 + \alpha \beta + \beta^2) = \text{roots of } x^3 + \frac{9}{2} x + \frac{27}{4} = 0 \]

\[ \text{Roots are } 6, 12, 3 \sqrt{3} \]

\[ \text{(ii)} \]

\[ k = \alpha \beta + \beta \gamma + \gamma \alpha \]

\[ = 6 \times 12 + 12 \times 3 + 3 \times 6 \]

\[ = 126 \]  

\[ \text{(iii)} \]

\[ \text{Substitute } x = y + 1 \]

\[ y^3 - 3y - 7 = 0 \]

\[ y^3 = 2y + 7 \]

\[ \Rightarrow S_3 = 2S_1 + 3 = 2 \times 0 + 7 \times 3 \]

\[ = 21 \]  

\[ \text{6(iii)} \]

\[ S_{-1} = \frac{1}{3x-1} + \frac{1}{3y-1} + \frac{1}{3z-1} \]

\[ = \frac{(3x-1)(3y-1)(3z-1)}{(3x-1)(3y-1)(3z-1)} \]

\[ \Rightarrow \]

\[ S_{-2} = S_1 - 2S_{-1} \]

\[ S_{-2} = \frac{1}{7} (0 - 2(\frac{2}{7})) \]

\[ = \frac{y}{49} \]  

\[ \Rightarrow \]

\[ \alpha + 2\alpha + 4\alpha = b \Rightarrow \beta = 7 \alpha \]  

\[ \Rightarrow \]

\[ \frac{1}{\beta^2} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-27}{49} \]  

\[ \Rightarrow 2 \beta^2 \alpha + 9 \beta - \frac{27}{49} = 0 \]  

\[ \Rightarrow \]

\[ \frac{1}{\beta^2} \cdot \frac{\beta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{-27}{49} \]  

\[ \Rightarrow \]

\[ (\alpha^2 + \alpha \beta + \beta^2) = \text{roots of } x^3 + \frac{9}{2} x + \frac{27}{4} = 0 \]

\[ \text{Roots are } 6, 12, 3 \sqrt{3} \]

\[ \text{(ii)} \]

\[ k = \alpha \beta + \beta \gamma + \gamma \alpha \]

\[ = 6 \times 12 + 12 \times 3 + 3 \times 6 \]

\[ = 126 \]  

\[ \text{(iii)} \]

\[ \text{Substitute } x = y + 1 \]

\[ y^3 - 3y - 7 = 0 \]

\[ y^3 = 2y + 7 \]

\[ \Rightarrow S_3 = 2S_1 + 3 = 2 \times 0 + 7 \times 3 \]

\[ = 21 \]
9. (i) \( \alpha + \beta + \gamma = 5, \alpha \beta + \beta \gamma + \gamma \alpha = 13 \)
\( \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha) \)
\( = 5^2 - 2 \times 13 = -1 \)

(ii) \( \beta = (\alpha \beta + \beta \gamma + \gamma \alpha) - 1 \times \alpha \beta \gamma = -1 \times 13 = -5 \times \gamma \)
\( \alpha \beta \gamma = 5 \times 13 \)
\( \Rightarrow \alpha \beta \gamma = 65 \)

10. \( \alpha \beta = (\alpha \beta)^2 - \alpha \beta^2 \)
\( = (1)^2 - 29 = -28 \)
\( \Rightarrow \alpha \beta = -14 \)

11. \( y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}} \)
\( \Rightarrow \frac{1}{\sqrt{y}} + \frac{9}{y} - 3 = 0 \)
\( \Rightarrow \frac{1}{\sqrt{y}} = 3 - \frac{9}{y} \)
\( \Rightarrow \frac{1}{\sqrt{y}} = 3 - \frac{9}{y} \)
\( \Rightarrow \frac{1}{\sqrt{y}} = 3 - \frac{9}{y} \)
\( \Rightarrow \alpha \beta = -3 \)

12. \( \frac{y}{1+y^3} + \frac{9}{y} - 3 = 0 \)
\( \Rightarrow \frac{y}{1+y^3} + \frac{9}{y} = 3 \)
\( \Rightarrow \frac{y}{1+y^3} = 3 - \frac{9}{y} \)

13. \( S_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow S_4 = \sum_{n=1}^{4} \frac{1}{n^2} \)
\( \Rightarrow S_4 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \)
\( \Rightarrow S_4 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \)
\( \Rightarrow S_4 = \frac{20}{4} + \frac{4}{4} + \frac{4}{4} + \frac{5}{4} \)
\( \Rightarrow S_4 = \frac{33}{4} \)

14. \( \alpha \beta + \beta \gamma + \gamma \alpha = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow \alpha \beta + \beta \gamma + \gamma \alpha = \frac{33}{4} \)

15. \( \alpha \beta \gamma = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow \alpha \beta \gamma = \frac{33}{4} \)

16. \( \alpha \beta \gamma = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow \alpha \beta \gamma = \frac{33}{4} \)

17. \( \alpha \beta \gamma = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow \alpha \beta \gamma = \frac{33}{4} \)

18. \( \alpha \beta \gamma = \sum_{n=1}^{\infty} \frac{1}{n^2} \)
\( \Rightarrow \alpha \beta \gamma = \frac{33}{4} \)
15. (i) \[
\frac{1}{\overline{\alpha \beta}(\overline{\alpha \beta})^2} = \frac{1}{(\overline{\alpha \beta})^2} = \frac{1}{\overline{\alpha \beta}} = \frac{1}{1} = 9
\]

(ii) \[
\frac{1}{\alpha \overline{\beta}} + \frac{1}{\overline{\alpha \beta}} = \frac{\overline{\alpha} \overline{\beta} + \alpha \beta}{\alpha \overline{\beta}(\overline{\alpha \beta})^2} = \frac{\overline{\alpha} \overline{\beta} + \alpha \beta}{9} \]

(iii) \[
\frac{1}{\beta} + \frac{1}{\alpha \overline{\beta}} = \frac{\overline{\alpha} \beta + \alpha \beta}{\alpha \overline{\beta}(\overline{\alpha \beta})^2} = \frac{\overline{\alpha} \beta + \alpha \beta}{9} \]

(iv) \[
\frac{1}{\alpha \overline{\beta}} + \frac{1}{\overline{\alpha \beta}} = \frac{\alpha \beta + \overline{\alpha \beta}}{\alpha \overline{\beta}(\overline{\alpha \beta})^2} = \frac{\alpha \beta + \overline{\alpha \beta}}{9} \]

(v) \[
\frac{1}{\alpha \overline{\beta}} + \frac{1}{\overline{\alpha \beta}} = \frac{\alpha \beta + \overline{\alpha \beta}}{\alpha \overline{\beta}(\overline{\alpha \beta})^2} = \frac{\alpha \beta + \overline{\alpha \beta}}{9} \]

(vi) \[
\frac{1}{\alpha \overline{\beta}} + \frac{1}{\overline{\alpha \beta}} = \frac{\alpha \beta + \overline{\alpha \beta}}{\alpha \overline{\beta}(\overline{\alpha \beta})^2} = \frac{\alpha \beta + \overline{\alpha \beta}}{9} \]

16. (i) \[
2 \alpha + 2 \beta = 0 \Rightarrow \beta = -\alpha
\]

(ii) \[
\alpha^2 + 4 \alpha \beta + \beta^2 = -9
\]

(iii) \[
2 \alpha^2 + 2 \alpha \beta^2 = -9
\]

(iv) \[
\alpha^2 \beta^2 = -2
\]

(v) \[
\beta = -\alpha \Rightarrow (i) \quad (ii) \quad (iii) \quad (iv)
\]

\[
\Rightarrow \beta = 2 \alpha^2 \quad \alpha \beta = -2
\]

\[
\Rightarrow \beta^2 + 4 \alpha = 0
\]

\[
\text{and} \quad \beta = 0
\]

17. \[
\alpha + \beta + \gamma = -\beta + 15 \Rightarrow \beta = -15
\]

\[
\alpha^2 + \beta^2 + \gamma^2 = (\overline{\alpha \beta})^2 - 2 \alpha \beta
\]

\[
\beta^2 = (-15)^2 - 2 \alpha
\]

\[
\Rightarrow \alpha = 77
\]

\[
\alpha \beta + \alpha \gamma = 36 \quad \text{(i)}
\]

\[
\beta + \gamma = 36
\]

\[
\alpha = 15 - \frac{36}{\alpha}
\]

\[
\alpha + \beta + \gamma = 15
\]

\[
\frac{15 - \alpha}{\alpha} \left( \begin{array}{c}
\alpha + \beta + \gamma = 15 \\
\beta + \gamma = 15
\end{array} \right)
\]

\[
\Rightarrow \alpha = 3
\]

\[
\alpha \beta = \alpha \beta + \beta \gamma + \gamma \alpha = 9 \quad \left[ \alpha = \frac{12}{3} \right]
\]

\[
\Rightarrow \beta = 3 \Rightarrow \gamma = -\alpha \beta = -3 \times 3 = -9
\]

\[
\Rightarrow \beta y = 35 \Rightarrow \lambda = -\alpha \beta \gamma = -3 \times 35 = -105 \Rightarrow \lambda = -105
\]