Further
Pure Math I

Roots of Polynomials

Notes

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Roots of Polynomials

Notes

Formulae & Algebra:

1. \((a+b)^2 = a^2 + b^2 + 2ab \Rightarrow a^2 + b^2 = (a+b)^2 - 2ab\)

2. \((a-b)^2 = a^2 + b^2 - 2ab \Rightarrow a^2 + b^2 = (a-b)^2 + 4ab\)

3. \((a+b)^3 = a^3 + b^3 + 3ab(a+b) \Rightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b)\)
   \[= (a+b)(a^2 + b^2 - ab)\]

4. \((a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \Rightarrow a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab + bc + ca)\)

5. \((a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b+c)(ab + bc + ca) - 3abc\)
   \[\Rightarrow a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b+c)(ab + bc + ca) + 3abc,\]
   \[\text{or } a^3 + b^3 + c^3 = (a+b+c)[a^2 + b^2 + c^2 - (ab + bc + ca)] + 3abc,\]
   \[= (a+b+c)[(a+b+c)^2 - 3(ab + bc + ca)] + 3abc.\]

6. \(a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2[(ab + bc + ca)^2 + (a^2b^2 + b^2c^2 + c^2a^2) - 2(a^2b^2 + b^2c^2 + c^2a^2)]\)
   \[= (a^2 + b^2 + c^2)^2 - 2[(ab + bc + ca)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) + 2abc(a+b+c)\]
   \[= [(a+b+c)^2 - 3(ab + bc + ca)]^2 - 2[(ab + bc + ca)^2 - 2abc(a+b+c).\]

Note 1. Symmetric functions of the roots of a polynomial equation:

(i) \(\alpha + \beta, \alpha^2 + \beta^2, \alpha \beta + \beta \alpha, \text{ are all symmetric functions for if } \alpha \text{ and } \beta \text{ are interchanged the functions remain the same.}\)

(ii) \(\alpha^2 + \beta^2 + \gamma^2 \text{ or } \alpha \beta + \beta \gamma + \gamma \alpha \text{ are also symmetric.}\)

Note 2. Let \(\alpha, \beta, \gamma, \cdots\) are the roots of a polynomial equation and \(S_n\) denotes,

\[S_n = \alpha^n + \beta^n + \gamma^n + \cdots\]

Then \(S_1 = \alpha + \beta + \gamma \cdots = \sum \alpha\)

\[S_2 = \alpha^2 + \beta^2 + \gamma^2 + \cdots = \sum \alpha^2\]

\[S_3 = \alpha^3 + \beta^3 + \gamma^3 + \cdots = \sum \alpha^3\]
Notes

Polynomial: 
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]

is polynomial of degree \( n \).

Here \( n \) is a positive integer; \( x \) is a variable.
\( a_0, a_1, a_2, \ldots \) are constant real numbers.

Polynomial Equation of degree \( n \):

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

A polynomial equation of degree \( n \) has at most \( n \) real roots.

***Linear Equation***: An equation of deg. 1.
\[ ax + b = 0 \]
\( a \neq 0 \) in linear equation, \( a \neq 0 \)
Example: \( 2x + 3 = 0 \)

***Quadratic Equation***: "degree 2"

\[ ax^2 + bx + c = 0, \ a \neq 0 \]
Example: \( 2x^2 + 5x - 7 = 0 \); \( x^2 - 3 = 0 \), \( 5x^2 = 0 \)

***Cubic Equation***: "degree 3"

\[ ax^3 + bx^2 + cx + d = 0, \ a \neq 0 \]
Example: \( 5x^3 - 3x^2 + x - 7 = 0 \)

***Quartic Equation (or Biquadratic Eqn.)***:

\[ ax^4 + bx^3 + cx^2 + dx + e = 0 \]; deg. = 4, \( a \neq 0 \)
Example: \( \frac{3}{4} x^4 - 5x^3 + 2x + 3 = 0 \)

& Roots of Polynomial Equation:

\( \alpha \) is root of a polynomial equation if \( \alpha \) is replaced by \( x \), makes it true.
Example: \( 3 \) is a root of \( x^2 - 5x + 6 = 0 \)
\( 3^2 - 5 \cdot 3 + 6 = 0 \) is true, \( \checkmark \)
Root of Polynomials

Notes

- Relation between the roots and coefficients of terms of a Quadratic Equation:

  Given a quadratic equation \( ax^2 + bx + c = 0 \)  \( \text{---(1)} \)

  Let \( \alpha \) and \( \beta \) are the roots of \( 0 \)

  \[ a(x-\alpha)(x-\beta) = ax^2 + bx + c \]

  \[ \alpha + \beta = -\frac{b}{a} \quad \text{---(2)} \]

  \[ \alpha \beta = \frac{c}{a} \quad \text{---(3)} \]

- Alternatively:

  Using quadratic formula: \( x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \)

  Let \( \alpha = \frac{-b + \sqrt{b^2-4ac}}{2a} \quad \text{---(i')} \)

  \[ \beta = \frac{-b - \sqrt{b^2-4ac}}{2a} \quad \text{---(ii')} \]

  Add (i') + (ii') \( \alpha + \beta = -\frac{b}{a} \quad \checkmark \)

  Multiply (i') \( \alpha \beta = \frac{c}{a} \quad \checkmark \)

- Example 1:

  Given a quadratic equation \( 2x^2 - 5x + 3 = 0 \)

  Find the sum and product of its roots.

  Solution: \( a = 2, \ b = -5, \ c = 3 \)

  Sum of roots \( \alpha + \beta = -\frac{b}{a} = -\frac{-5}{2} = \frac{5}{2} \quad \checkmark \)

  Product of roots \( \alpha \beta = \frac{c}{a} = \frac{3}{2} \quad \checkmark \)

  Alternate method:

  \( 2x^2 - 5x + 3 = 0 \)

  \( \Rightarrow (x-1)(2x-3) = 0 \)

  Roots are \( 1, \frac{3}{2} \)

  Sum = \( 1 + \frac{3}{2} = \frac{5}{2} \quad \checkmark \)

  Product of roots \( = 1 \times \frac{3}{2} = \frac{3}{2} \quad \checkmark \)
Example 2 (conversely) Find a quadratic equation whose roots \( \alpha \) and \( \beta \) are given as 7 and -4 respectively.

(ii) Form another quad. equation whose roots are \( \alpha + 1 \) and \( \beta + 1 \).

Solution: (i) Sum of roots \( \alpha + \beta = 7 + (-4) = 3 \)

\[ x^2 - (\alpha + \beta)x + \alpha \beta = 0 \]

Product of roots \( \alpha \beta = 7 \times (-4) = -28 \)

\[ x^2 - 3x + (-28) = 0 \]

from (i) (ii)

or \[ x^2 - 3x - 28 = 0 \checkmark \]

(ii) Now given new roots \( \alpha + 1 \) and \( \beta + 1 \)

Sum of new roots = \( (\alpha + 1) + (\beta + 1) \)

\[ = (\alpha + \beta) + 2 = 3 + 2 = 5 \]

and Product of new roots = \( (\alpha + 1)(\beta + 1) \)

\[ = (\alpha + \beta) + \alpha \beta + 1 = 3 + (-28) + 1 \]

\[ = -24 \]

(iii) Reqd. quad. eqn

\[ x^2 - (\text{Sum of new roots})x + \text{Product of new roots} = 0 \]

or \[ x^2 - 5x - 24 = 0 \]

Verify:

(iii) New roots are \( \alpha^2 \) and \( \beta^2 \)

Sum of new roots = \( \alpha^2 + \beta^2 \)

\[ = (\alpha + \beta)^2 - 2\alpha \beta \quad \text{from (i)} \]

\[ = 3^2 - 2 \times (-28) = 65 \]

Product of new roots = \( \alpha^2 \beta^2 = (\alpha \beta)^2 \)

\[ = (-28)^2 = 784 \]

(iii) Reqd. quad. eqn

\[ x^2 - (\text{Sum of new roots})x + \text{Product} = 0 \]

or \[ x^2 - 65x + 784 = 0 \]

(ii) Alt: New roots \( \alpha^2, \beta^2 \) are \( 7^2 \leq (-4)^2 \) or \( 49 \leq 16 \),

Sum = 65 and Prod = 784 \[ \therefore x^2 - 6.5x + 784 = 0 \checkmark \]
Forming new equations using "Substitution method".

Example 3. Given a quadratic equation \( x^2 - 3x - 28 = 0 \)
has roots \( \alpha \) and \( \beta \), form a new quad. equation
(i) whose roots are \( \alpha^2 \) and \( \beta^2 \) (ii) roots are \( \alpha+1 \) and \( \beta+1 \)

Solution: Give quad eqn \( x^2 - 3x - 28 = 0 \) --- \( \Omega \)

Given roots are \( \alpha \) and \( \beta \),

(i) To form a new eqn whose roots are \( \alpha^2 \) and \( \beta^2 \)

Let \( y = \alpha^2 \) \( \Rightarrow \) \( x = \sqrt{y} \) in eqn \( \Omega \)

we get, \( (\sqrt{y})^2 - 3\sqrt{y} - 28 = 0 \)

\( \Rightarrow \) \( y - 3\sqrt{y} - 28 = 0 \)

\( \Rightarrow \) \( 3\sqrt{y} = 28 - y \)

squaring both sides \( 9y = (28-y)^2 \)

\( \Rightarrow \) \( 9y = 784 + y^2 - 56y \)

\( \Rightarrow \) \( y^2 - 65y - 784 = 0 \) \( \text{[Note: Same as example 2 (i)]} \)

(ii) New roots are \( \alpha+1 \) and \( \beta+1 \)

put \( y = x+1 \) \( \Rightarrow \) \( x = y-1 \)

but \( x = \sqrt{y} \) in eqn \( \Omega \)

\( (y-1)^2 - 3(y-1) - 28 = 0 \) \( \Rightarrow \) \( y^2 - 2y + 1 - 3y + 3 - 28 = 0 \)

\( \Rightarrow \) \( y^2 - 5y - 24 = 0 \) \( \text{[Note: Same as example 2 (i)]} \)

we may replace \( y \) by \( x \),

Required equation is

\( x^2 - 5x - 24 = 0 \) \( \checkmark \)
Cubic Equations:

General form: \( a\alpha^3 + b\beta^2 + c\gamma + d = 0 \) — (1)

Let \( \alpha, \beta \) and \( \gamma \) are roots of equation.

\( \Rightarrow a(\alpha - \alpha)(\beta - \beta)(\gamma - \gamma) = a\alpha^3 + b\beta^2 + c\gamma + d \)

\( \equiv [\alpha^2 - (\alpha + \beta + \gamma)\alpha + (\alpha\beta + \beta\gamma + \gamma\alpha)\alpha - \alpha\beta\gamma] = 0 \)

Comparing the coefficients on both sides, we get:

\[ \begin{align*}
\sum \alpha &= -\frac{b}{a} \quad \text{— (i)} \quad \checkmark \\
\sum \alpha \beta &= \frac{c}{a} \quad \text{— (ii)} \quad \checkmark \\
\text{and} \sum \alpha \beta \gamma &= -\frac{d}{a} \quad \text{— (iii)} \quad \checkmark
\end{align*} \]

Conversely:

Given \( \alpha, \beta, \gamma \) are the roots of a cubic equation,

Required cubic equation is:

\[ x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0 \]

Note: \( \sum \alpha^2 \beta = (\alpha^2 + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) \)

Example 4: The cubic equation \( 2\alpha^3 - 3\alpha^2 + 4\alpha - 10 = 0 \) has roots \( \alpha, \beta \) and \( \gamma \). (i) Find the value of \( (\alpha + 1)(\beta + 1)(\gamma + 1) \)

Solution:

For the given cubic equation, \( \sum \alpha + \beta + \gamma = -\frac{b}{a} = \frac{3}{2} \)

\( (i) \) \( (\alpha + 1)(\beta + 1)(\gamma + 1) \)

\( = (\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma + 1 \)

\( = \frac{3}{2} + \frac{4}{2} + 1 = \frac{5}{2} \quad \text{from (i)} \)

\( (ii) \) \( (\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) \)

\( \frac{3}{2} - \alpha)(\frac{3}{2} - \beta)(\frac{3}{2} - \gamma) \quad \text{from (i)} \)

\( = \frac{27}{8} - \frac{9}{4} (\alpha + \beta + \gamma) - \alpha\beta\gamma + \frac{3}{2} (\alpha\beta + \beta\gamma + \gamma\alpha) \)

\( \text{from (i), (ii), (iii)} \)

\( \frac{27}{8} - \frac{9}{4} \times \frac{3}{2} - 5 + \frac{3}{2} \times \frac{3}{2} = -\frac{27}{8} \)
Example 5: The cubic equation \( Z^3 - Z^2 - Z - 5 = 0 \) has roots \( \alpha, \beta \) and \( \gamma \).

(a) Show that the value of \( \alpha^3 + \beta^3 + \gamma^3 \) is 19. \(-[4]\)

(b) Find the value of \( \alpha^4 + \beta^4 + \gamma^4 \). \(-[2]\)

(c) Find a cubic equation with roots \( \alpha + 1, \beta + 1 \) and \( \gamma + 1 \), give your answer in the form \( bx^3 + 9x^2 + 2x \cdot d = 0 \) where \( b, q, r \) and \( d \) are constants to be determined. \(-[3]\)

Solution:

Given \( Z^3 - Z^2 - Z - 5 = 0 \) \(-[7]\)

\( \alpha + \beta + \gamma = (-1) = 1; \quad \alpha \beta + \beta \gamma + \gamma \alpha = (-5) = 5 \\checkmark \)

\( \alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha \beta - \beta \gamma - \gamma \alpha) + 3\alpha \beta \gamma \)

\( = (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha \beta + \beta \gamma + \gamma \alpha)] + 3\alpha \beta \gamma \)

\( = 1 \left[ 1^2 - 3(-1) \right] + 3 \times 5 = 19 \checkmark \)

Alternate method: from \( \theta \)

\( Z^3 = Z^2 + Z + 5 \)

\( \Rightarrow \alpha^3 + \beta^3 + \gamma^3 = \sum \alpha^3 = \sum \alpha \cdot \sum \alpha^2 + \sum \alpha = \left[ (\alpha + \beta + \gamma)^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha) \right] + (\alpha + \beta + \gamma) + 5 \times 3 \)

\( = [1^2 - 2 \times (-1)] + 1 + 15 = 19 \checkmark \)

(b) \( \alpha^4 + \beta^4 + \gamma^4 = \left[ (\sum \alpha)^2 - 2 \sum \alpha \beta \right] - \left[ (\sum \alpha)^2 - 2 \sum \alpha \beta \right] \)

\( = \left[ 1^2 - 2 \times (-1) \right]^2 \left[ 1^3 - 2 \times 1 \times 1 \right] \quad \text{On Page 1} \)

\( = (3)^2 - 2 \times (-9) = \frac{37}{\text{formula} 6} \)

Alternate method:

from \( \theta \) \( Z^3 = Z^2 + Z + 5 \Rightarrow Z^4 = Z^3 + Z^2 + 5Z \)

\( \Rightarrow \alpha^4 + \beta^4 + \gamma^4 = \sum \alpha^4 = \sum \alpha^3 + \sum \alpha^2 + 5 \sum \alpha \)

Consider \( \sum \alpha^2 = (\sum \alpha)^2 - 2 \sum \alpha \beta \)

\( = \left[ \sum \alpha \right]^2 - 2 \times \sum \alpha \beta = 3 - 3 \) \( \checkmark \) \( \text{from} \quad \theta \quad \text{and} \quad \gamma \)

(c) New roots are \((x + 1), (y + 1)\) and \((z + 1)\).

Let \( x = z + 1 \Rightarrow z = (x - 1) \) in \( \theta \)

\( \Rightarrow (x - 1)^3 - (x - 1)^2 - (x - 1) - 5 = 0 \)

\( \Rightarrow x^3 - 4x^2 + 4x - 6 = 0 \checkmark \)
Example 6: Find the cubic equation with roots \( \alpha, \beta \) and \( \gamma \), such that:

\[
\begin{align*}
\alpha + \beta + \gamma &= 3 \quad (1) \\
\alpha^2 + \beta^2 + \gamma^2 &= 1 \quad (2) \\
\alpha^3 + \beta^3 + \gamma^3 &= -30 \quad (3)
\end{align*}
\]

giving your answer in the form \( x^3 + px^2 + qx + r = 0 \) where \( p, q, \) and \( r \) are integers to be found. \([W-1611182](W-1611182)\)

Solution: The required cubic eqn will be:

\[
x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha \beta + \beta \gamma + \gamma \alpha)x - \alpha \beta \gamma = 0
\]

Now \( \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha) \)

\[
\begin{align*}
\text{from } (1) & \quad 1 = 3^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha) \\
\Rightarrow & \quad (\alpha \beta + \beta \gamma + \gamma \alpha) = 4 \quad (5)
\end{align*}
\]

We know:

\[
(\alpha^3 + \beta^3 + \gamma^3) - 3\alpha \beta \gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha \beta - \beta \gamma - \gamma \alpha)
\]

\[
\begin{align*}
\text{from } (3), (5) & \quad -30 - 3\alpha \beta \gamma = 3[1 - 4] \\
\Rightarrow & \quad \alpha \beta \gamma = -7 \quad (6)
\end{align*}
\]

\[
\text{from } (6) \text{ Rep. cubic eqn: }
\]

\[
x^3 - 3x^2 + 7x + 7 = 0
\]

Quartic Equation (or Biquadratic Equation):

\[
a x^4 + b x^3 + c x^2 + d x + e = 0
\]

Let the roots of this quartic equation are \( \alpha, \beta, \gamma \) and \( \delta \)

Then:

\[
x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha \beta + \beta \gamma + \gamma \delta + \delta \alpha)x^2 - (\alpha \beta \gamma + \alpha \gamma \delta + \beta \gamma \delta + \delta \alpha \beta)x + \alpha \beta \gamma \delta = 0
\]

\[
\begin{align*}
\Delta & = -\frac{b}{a} \\
\Delta & = \frac{c}{a} \\
\Delta & = -\frac{d}{a} \\
\Delta & = \frac{e}{a}
\end{align*}
\]

and \( \alpha \beta \gamma \delta = \frac{e}{a} \)
Example 7: The roots of the quartic equation,
\[ 2x^4 + 4x^3 + 2x^2 - 4x + 1 = 0 \]
are \( \alpha, \beta, \gamma, \) and \( \delta. \)

Find the value of:
(i) \( \alpha + \beta + \gamma + \delta \)
(ii) \( \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \)
(iii) \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} \)
(iv) \( \frac{\alpha}{\beta + \delta} + \frac{\beta}{\alpha + \gamma} + \frac{\gamma}{\beta + \delta} + \frac{\delta}{\alpha + \gamma} \)

Using substitution \( y = x + 1, \) find a quartic
equation in \( y. \) Solve this quartic equation, and hence
find the roots of the equation,
\[ 2y^4 + 4y^3 + 2y^2 - 4y + 1 = 0. \]

Solution:
(i) \( \alpha + \beta + \gamma + \delta = -\frac{b}{a} = -4 \)

(ii) \( \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \left( \alpha + \beta + \gamma + \delta \right)^2 - 2(\alpha \beta + \beta \gamma + \gamma \delta + \delta \alpha) = (-4)^2 - 2 \times (-4) = 12 \)

(iii) \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha \beta + \beta \gamma + \gamma \delta + \delta \alpha}{\alpha \beta + \beta \gamma + \gamma \delta + \delta \alpha} = \frac{12}{12} = 1 \)

(iv) \( \frac{\alpha}{\beta + \delta} + \frac{\beta}{\alpha + \gamma} + \frac{\gamma}{\beta + \delta} + \frac{\delta}{\alpha + \gamma} \)

Now let \( y = x + 1 \) \( \Rightarrow \) \( x = y - 1 \) but in (i)
\( (y-1)^4 + 4(y-1)^3 + 2(y-1)^2 - 4(y-1) + 1 = 0 \)
\( \Rightarrow \left( y^4 - 4y^3 + 6y^2 - 4y + 1 \right) + \left( y^3 - 3y^2 + 3y - 1 \right) = 0 \)
\( \Rightarrow y^4 - 4y^2 + 4 = 0 \)
\( \Rightarrow (y^2 - 2)^2 = 0 \)
\( \Rightarrow y^2 = 2 \) \( \Rightarrow \) \( y = \pm \sqrt{2} \)

\( \therefore \) Roots of the new quartic equation (ii) are:
\( \sqrt{2}, -\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \)

Now \( x = y - 1 \) so the roots of the original equ. (i) are:
\( \sqrt{2} - 1, -\sqrt{2} - 1, -\sqrt{2}, -\sqrt{2} - 1 \)
Example 8: The quartic equation $x^4 - px^2 + qx - r = 0$
where $p, q,$ and $r$ are real constants, has two pairs of equal roots, show that $p^3 + 4r = 0$ and state the value of $q.$

Solution: Let the roots are $\alpha, \alpha, \beta$ and $\beta.$

\[ \begin{align*}
\text{Sum of roots: } 2\alpha + 2\beta &= 0 \Rightarrow \beta = -\alpha \quad \text{(i)} \\
\Sigma \alpha \beta &= \alpha^2 + 4\alpha \beta + \beta^2 = -p \\
\Sigma \alpha \beta \gamma &= 2\alpha^2 \beta + 2\alpha \beta^2 = -q \\
\Sigma \alpha \beta \gamma \delta &= \alpha \beta \delta = -r
\end{align*} \]

From (i) put $\beta = -\alpha$ in (ii) and (iv)

\[ p = 2\alpha^2 \quad \text{and} \quad \alpha^4 = -r \Rightarrow p^3 + 4r = 0 \]

Also put $\beta = -\alpha$ in (iii)

\[-2\alpha^2 \beta^2 + 2\alpha \beta^2 = -q \Rightarrow q = 0 \]
**Roots of Polynomials**

- **Sum of integral power (equal) if the roots of a polynomial:**

  Given a polynomial of degree
  \[ a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \]

  Let the roots of this polynomial be \( \alpha, \beta, \gamma, \ldots \) and

  \[ S_n = \alpha^n + \beta^n + \gamma^n + \cdots \]

  \[ S_1 = \alpha + \beta + \gamma + \cdots \]

  \[ S_2 = \alpha^2 + \beta^2 + \gamma^2 + \cdots \]

  To find \( S_1, S_2, S_3, \ldots \), we may write form 1

  \[ a_0 S_1 + a_1 = 0 \Rightarrow S_1 = -\frac{a_1}{a_0} \]

  \[ a_0 S_2 + a_1 S_1 + a_2 = 0 \Rightarrow S_2 = -\frac{(a_1 S_1 + 2a_2)}{a_0} \]

  and

  \[ a_0 S_3 + a_1 S_2 + a_2 S_1 + a_3 = 0 \]

  Let the value of \( S_1, S_2, S_3 \) we get \( S_3 = 0 \)

- **Example 9:** Given \( \alpha, \beta, \gamma \), are the roots of a quad. eqn. \( x^2 - 5x + 6 = 0 \)

  Find (i) \( \alpha + \beta \)

  (ii) \( \alpha^2 + \beta^2 \)

  (iii) \( \alpha^3 + \beta^3 \)

  (iv) \( \alpha^4 + \beta^4 \)

  Solution:

  \[ S_2 - 5S_1 + 6 = 0 \]

  \[ \Rightarrow 2 \alpha = \alpha + \beta = S_1 = 5 \]

  From (i) \[ \alpha + \beta = -\frac{b}{a} = -\frac{5}{1} = -5 \]

  \[ \alpha \beta = \frac{c}{a} = \frac{6}{1} = 6 \]

  From (ii) \[ \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \beta = 5^2 - 2 \times 5 \times 6 = 25 - 60 = -35 \]

  From (iii) \[ \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha \beta + \beta^2) = -5(5 - 6) = -5 \times (-1) = 5 \]

  From (iv) \[ \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2 \beta^2 = 25^2 - 2 \times 6^2 = 625 - 72 = 553 \]

  To find \( S_3 \), multiply (i) by \( x^2 \)

  \[ x^3 - 5x^2 + 6x = 0 \]

  \[ \Rightarrow S_3 = 5S_2 - 6S_1 = 5 \times 3 - 6 \times 5 \]

  \[ = 15 - 30 = -15 \]

  To find \( S_4 \), multiply (i) by \( x^2 \)

  \[ x^4 - 5x^3 + 6x^2 = 0 \]

  \[ \Rightarrow S_4 - 5S_3 + 6S_2 = 0 \]

  \[ S_4 - 5 \times 35 + 6 \times 13 = 0 \Rightarrow S_4 = 97 \]
Example 10: The cubic equation \( z^3 - z^2 - z - 5 = 0 \) has roots \( \alpha, \beta \) and \( \gamma \). 

(a) Show that the value of \( \alpha^3 + \beta^3 + \gamma^3 \) is 19. 

(b) Find the value of \( \alpha^4 + \beta^4 + \gamma^4 \). 

Solution: 

Denote \( S_1 = \alpha + \beta + \gamma \) 

Given \( z^3 - z^2 - z - 5 = 0 \) \( \quad \) \( \text{(1)} \) 

\( S_1 - 1 = 0 \Rightarrow S_1 = 1 \) \( \quad \) \( \text{(2)} \) 

Again \( S_2 - S_1 + 2(-1) = 0 \quad S_3 = \alpha^3 + \beta^3 + \gamma^3 \) 

\( S_2 - 1 - 2 = 0 \Rightarrow S_2 = 3 \) \( \quad \) \( \text{(3)} \) 

Also \( S_3 - S_2 = 3(-5) = 0 \Rightarrow S_3 = 3 - 15 = 0 \) 

\( \Rightarrow S_3 = 19 \) \( \quad \) \( \text{(4)} \) 

Now to get \( S_4 \), multiply \( \text{(1)} \) by \( z \): 

\[ z^4 - z^3 - z^2 - 5z = 0 \] 

\( S_4 - S_3 - S_2 - 5S_1 = 0 \) 

\( \Rightarrow S_4 - 19 - 3 - 5 = 0 \quad S_4 = 27 \) \( \quad \) \( \text{(5)} \) 

Example 11: The roots of the cubic equation \( \alpha^3 - 5\alpha^2 + 13\alpha - 4 = 0 \) are \( \alpha, \beta, \gamma \). 

(i) Find \( \alpha^2 + \beta^2 + \gamma^2 \) 

(ii) Find the value of \( \alpha^3 + \beta^3 + \gamma^3 \) 

Solution: 

Given: \( \alpha^3 - 5\alpha^2 + 13\alpha - 4 = 0 \) \( \quad \) \( \text{(1)} \) 

\( S_1 = \alpha + \beta + \gamma \) 

\( S_2 = \alpha^2 + \beta^2 + \gamma^2 \) 

\( S_3 = \alpha^3 + \beta^3 + \gamma^3 \) 

\( S_1 - 5 = 0 \Rightarrow S_1 = 5 \) \( \quad \) \( \text{(2)} \) 

\( S_2 - 5S_1 + 132 = 0 \) 

\( \Rightarrow S_2 = 5S_1 - 132 = 0 \Rightarrow S_2 = 27 \) \( \quad \) \( \text{(3)} \) 

Again: \( S_3 = S_2 \) \( \quad \) \( \text{(4)} \) 

\( S_3 - 5S_2 + 135 - 4 \times 3 = 0 \) 

\( S_2 - 5 \times (-1) + 13 \times 5 - 12 = 0 \) 

\( \Rightarrow S_3 = -58 \) \( \quad \) \( \text{(5)} \)
Example 12. The roots of the equation \(x^4 - 3x^3 + 5x - 2 = 0\) are \(\alpha, \beta, \gamma\), and \(\delta\). 
and \(S_n\) denotes \(\alpha^n + \beta^n + \gamma^n + \delta^n\). 
Show that \(S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0\). 
Find the values of 
(i) \(S_2\) and \(S_4\) 
(ii) \(S_3\) and \(S_5\).

Hence find the value of: 
\[\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3)\]

Solution 
Given \(x^4 - 3x^3 + 5x - 2 = 0\) \(\text{Eqn. 1}\) 
Multiply \(\text{Eqn. 1}\) by \(x^n\) \(\Rightarrow x^{n+4} - 3x^{n+2} + 5x^{n+1} - 2x^n = 0\) \(\text{Eqn. 2}\)

\(\alpha^n\) is a root of \(\text{Eqn. 2}\) \(\Rightarrow (\alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0)\) \(\text{Eqn. 3}\)

\(\beta^n\) is a root of \(\text{Eqn. 2}\) \(\Rightarrow (\beta^{n+4} - 3\beta^{n+2} + 5\beta^{n+1} - 2\beta^n = 0)\) \(\text{Eqn. 4}\)

\(\gamma^n\) is a root of \(\text{Eqn. 2}\) \(\Rightarrow (\gamma^{n+4} - 3\gamma^{n+2} + 5\gamma^{n+1} - 2\gamma^n = 0)\) \(\text{Eqn. 5}\)

\(\delta^n\) is a root of \(\text{Eqn. 2}\) \(\Rightarrow (\delta^{n+4} - 3\delta^{n+2} + 5\delta^{n+1} - 2\delta^n = 0)\) \(\text{Eqn. 6}\)

Add \(\text{Eqn. 3, 4, 5, & 6}\) \(\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0\) \(\text{Eqn. 7}\)

(i) From \(\text{Eqn. 7}\) \(S_n - 0 = 0 \Rightarrow S_n = 0\). \(\text{Eqn. 8}\)

Also from \(\text{Eqn. 7}\) \(S_2 + 6S_1 - 3S_2 = 0 \Rightarrow S_2 - 6 = 0 \Rightarrow S_2 = 6\). \(\text{Eqn. 9}\)

Again from \(\text{Eqn. 7}\) put \(n = 0\): \(S_4 - 3S_2 + 12x4 = 0 \Rightarrow S_4 - 3x6 - 8 = 0\)
\(\Rightarrow S_4 = 26\). \(\text{Eqn. 10}\)

\(\text{i}\) \(S_3 + 0S_2 - 3S_1 + 5x3 = 0\)
\(\Rightarrow S_3 - 3x0 + 15 = 0 \Rightarrow S_3 = -15\). \(\text{Eqn. 11}\)

Multiply \(\text{Eqn. 7}\) by \(x^3\): \(x^5 - 3x^3 + 5x^2 - 2x = 0\)
\(\Rightarrow S_5 - 3S_3 + 5S_2 - 2S_1 = 0\)
\(\Rightarrow S_5 = 3(S_3 - 15) + 5x6 - 0 = 0\)
\(\Rightarrow S_5 = -75\). \(\text{Eqn. 12}\)

Now \(\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3)\)
\(\Rightarrow S_3 (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) - (\alpha^5 + \beta^5 + \gamma^5 + \delta^5)\)
\(\Rightarrow S_3 x S_2 - S_5\)
\(\Rightarrow -15x6 - (-75) = -90 + 75 = -15\).