Further
Pure Maths 1

Summation of Series
Notes

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Some Standard Results:

1. \[ \sum_{i=1}^{n} 1 = n \]

2. \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

3. \[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

4. \[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 = \frac{1}{4} n^2 (n+1)^2 \]

In General, \( S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \)

Partial Fractions:

Given an algebraic fraction, whose denominator is expressed as product of linear factors, and the numerator can be expressed as algebraic sum of functions with denominators as the factors, if given expression in \( D \).

Example 1: \( \frac{2 + y}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x+2} \)

Multiply each term by \( x(x+1)(x+2) \):

\[ 2 + y = a(x+1)(x+2) + b(x)(x+2) + c(x+1) \]

To get the value of \( a \), put \( x = 0 \) in \( 2 \):

\[ 2 = a(0+1)(0+2) \Rightarrow 2a = 2 \Rightarrow a = 1 \]

To get \( b \), put \( x = -1 \) (since \( x+1 \)):

\[ 2 + y = b(-1)(-2) \Rightarrow -b = 3 \Rightarrow b = -3 \]

And for \( c \), put \( x = -2 \) or \( x = -1 \):

\[ 2 + y = c \times 1 \times 0 \Rightarrow c = 2 \]

Now put the value of \( a \), \( b \) and \( c \) in \( 1 \):

The required partial fractions are:

\[ \frac{2 + y}{x(x+1)(x+2)} = \frac{1}{x} - \frac{3}{x+1} + \frac{2}{x+2} \]
Example: Use the method of difference to show that:

(i) \[ \sum_{i=1}^{N} \frac{1}{(3i+1)(3i-2)} = \frac{1}{3} - \frac{1}{3(3N+1)} \]

(ii) Find the limit as \( N \to \infty \), of

\[ \sum_{i=N+1}^{2N} \frac{N}{(3i+1)(3i-2)} \]

Solution: Consider (for partial fractions)

(i) \[ \frac{1}{(3i+1)(3i-2)} = \frac{a}{3i+1} + \frac{b}{3i-2} \]

From (1)

Put (zero of \( 3i+1 \)) \( i = -\frac{1}{3} \) in (1)

\[ 1 = a(3 \cdot \frac{1}{3} - 2) = -3a = 1 \Rightarrow a = -\frac{1}{3} \]

Put the values \( a = -\frac{1}{3} \) and \( b = \frac{1}{3} \) in (1)

\[ \frac{1}{(3i+1)(3i-2)} = \frac{1}{3} \left[ \frac{1}{3i-2} - \frac{1}{3i+1} \right] \]

\[ \sum_{i=1}^{N} \frac{1}{(3i+1)(3i-2)} = \frac{1}{3} \left[ \frac{1}{3 \cdot 3N-2} - \frac{1}{3N+1} \right] \]

\[ = \frac{1}{3} \left[ 1 - \frac{1}{3N+1} \right] = \frac{1}{3} - \frac{1}{3(3N+1)} \]

(ii) \[ \sum_{i=N+1}^{2N} \frac{N}{(3i+1)(3i-2)} = \frac{N^2}{3} \left[ \frac{1}{3N+1} - \frac{1}{3(3N+1)} \right] = \left( \frac{N^2}{3} - \frac{N}{3} \right) \left( \frac{1}{3N} \right) \left( \frac{3N+1}{3} \right) \]

(Continued...)
\[ F_p \]

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\[ 2(i) \quad \frac{N}{3(3N+1)} = \frac{N}{3(3N^2+1)} \]

\[ = \frac{N^3 - N^2}{(3N+1)(3N^2+1)} \]

\[ = 1 - \frac{1}{N} \]

Divide \( N^2 \) and \( B \)

by \( N^3 \)

\[ \frac{1}{N} \to 0 \]

\[ \frac{N}{3N^2} \to 0 \]

\[ = \frac{1}{9} \]

As \( n \to \infty \)

\[ \frac{1}{N} \to 0 \]

\[ \frac{N}{3N^2} \to 0 \]

\[ = \frac{1}{9} \]

Example 3

Let \( f(x) = 3 \cdot (x+1)(x+2) \) show that

\[ f(x) - f(x-1) = 3 \cdot (x+1) \]

Hence show that

\[ \sum_{x=1}^{n} x(x+1) = \frac{n}{3} \cdot (n+1)(n+2) \]

Using the standard results for \( \sum_{x=1}^{n} x \), deduce that

\[ \sum_{x=1}^{n} x^2 = \frac{1}{6} \cdot n(n+1)(2n+1) \]

Find the sum of the series

\[ 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \ldots + n^2 \]

where \( n \) is odd.

\[ \left( \frac{n-1}{2} \right)^2 \cdot n \cdot \left( n+1 \right) \]

Solution:

\[ f(x) = x(x+1)(x+2) \]

\[ f(x) - f(x-1) = x(x+1)(x+2) - (x-1)x(x+1) \]

\[ = x(x+1)[(x+2) - (x-1)] = 3 \cdot x(x+1) \]

Now

\[ \sum_{x=1}^{n} x(x+1) = \frac{n}{3} \cdot \frac{n(n+1)(2n+1)}{2} \]

\[ = \frac{1}{3} \cdot \frac{n(n+1)(n+2)}{2} \]

\[ = \frac{1}{3} \cdot \frac{n(n+1)(n+2)}{2} \]

\[ = \frac{1}{3} \cdot \frac{n(n+1)(n+2)}{2} \]

From (3)

\[ \frac{n(n+1)(n+2)}{2} \]

(continued ->)
3. Now to find the sum of the series,

\[
\sum_{k=1}^{n} \left( k^2 + 2k^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \cdots + 2(n-1)^2 + n^2 \right)
\]

\[
= \left( \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k^2 \right) \text{ (n is odd)}
\]

\[
= \frac{n(n+1)(2n+1)}{6} + 4 \times \frac{(n-1)(n)(2n+1)}{3} \times n \quad \Rightarrow (n-1) \text{ is even}
\]

\[
= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)(n-1)}{6}
\]

\[
= \frac{n(n+1)}{6} \left[ (2n+1) + (n-1) \right]
\]

\[
= \frac{n(n+1)}{6} \times 3n = \frac{1}{6} n^2 (n+1) \sqrt{n}
\]

Example: It is given that, \( S_n = \sum_{k=1}^{n} U_n = 2n^2 + n \)

Write down the values of \( S_1, S_2, S_3, S_4 \). Express \( U_n \) in terms of \( n \). Justify your answer.

Find \( \sum_{n=1}^{2n} U_n \).

Solution: Given \( S_n = \sum_{k=1}^{n} U_n = 2n^2 + n \) \( \quad (1) \)

From (1) \( S_1 = 2 \times 1^2 + 1 = 3 \)

\( S_2 = 2 \times 2^2 + 2 = 10 \quad \text{and} \quad S_3 = 2 \times 3^2 + 3 = 21 \)

\( \text{and} \quad S_4 = 2 \times 4^2 + 4 = 36 \)

Now \( U_n = \frac{2n}{2n-1} U_n = S_n - S_{n-1} \)

\( \Rightarrow \frac{2n}{2n-1} U_n = (2n^2 + n) - \left( 2(n-1)^2 + (n-1) \right) \)

And \( \sum_{n=1}^{2n} U_n = S_{2n} - S_n = \left( (2 \times 2n^2 + 2n) - (2n^2 + n) \right) \)

\( = 6n + n \)
Example 5: Given that \[ u_k = \frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k+1}} \]

Express \( \sum_{k=13}^{\infty} u_k \) in terms of \( n \), and deduce the value \( \sum_{k=13}^{\infty} u_k \).

Solution: Given \( u_k = \frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k+1}} \)

\[ \sum_{k=13}^{n} u_k = \left[ \frac{1}{\sqrt{25}} - \frac{1}{\sqrt{27}} \right] + \frac{1}{\sqrt{27}} - \frac{1}{\sqrt{29}} = \frac{1}{\sqrt{29}} \]

Add: \( \frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n+1}} \)

\[ \sum_{k=13}^{n} u_k = \frac{1}{5} - \frac{1}{\sqrt{2n+1}} \]

\[ \sum_{k=13}^{\infty} u_k = \lim_{n \to \infty} \left( \frac{1}{5} - \frac{1}{\sqrt{2n+1}} \right) = \frac{1}{5} \]

Example 6: Express \( \frac{4}{(x+1)(x+2)} \) into partial fractions and find \( \sum_{x=1}^{n} \frac{4}{x(x+1)(x+2)} \)

Deduce the value of \( \sum_{x=1}^{\infty} \frac{4}{x(x+1)(x+2)} \).

Solution: \( \frac{4}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x+2} \) for partial fractions.

Multiplying by \( x(x+1)(x+2) \), we get:

\[ 4 = a(x+1)(x+2) + b(x+2) + c(x+1) \quad (\text{1}) \]

But \( x=0 \) in (1) \( \Rightarrow 4 = 2a \Rightarrow a = 2 \)

Now \( x+1=0 \) \( \Rightarrow x = -1 \) in (2) \( \Rightarrow 4 = 0 + b(-1)(1) \Rightarrow b = -4 \)

And \( x+2=0 \) \( \Rightarrow x = -2 \) in (2) \( \Rightarrow 4 = 0 + c(-2)(-1) \Rightarrow c = 2 \)

From (1) Rep. partial fractions:

\[ \frac{4}{x} = 2 \cdot \frac{1}{x+1} + \frac{2}{x+2} \]

(continued →)
Example 6: \( \sum_{n=1}^{\infty} \frac{4}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \left( \frac{2}{n} - \frac{3}{n+1} + \frac{3}{n+2} \right) \)

\[
\begin{align*}
&= \left[ \frac{2}{1} - \frac{3}{2} + \frac{3}{3} \\
&\quad+ \frac{2}{2} - \frac{4}{3} + \frac{3}{4} \\
&\quad+ \frac{2}{3} - \frac{5}{4} + \frac{3}{5} \\
&\quad+ \frac{2}{4} - \frac{6}{5} + \frac{3}{6} \\
&\quad+ \cdots \right] \\
&= 1 - \frac{2}{n+1} + \frac{2}{n+2} \\
\end{align*}
\]

Now, \( \sum_{n=1}^{\infty} \frac{4}{n(n+1)(n+2)} \)

\[
\begin{align*}
&= \lim_{n \to \infty} \left( 1 - \frac{2}{n+1} + \frac{2}{n+2} \right) \approx 1 - 0 + 0 \\
&= 1
\end{align*}
\]