Further Pure Maths 2

Integration Exercise

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Integration

1. Find the exact value of \( \int_0^1 \frac{1}{\sqrt{3+4x-4x^2}} \, dx \). [6]

2. The diagram shows the curve with equation \( y = \frac{1}{x^2} \) for \( x > 0 \), together with a set of \((n-1)\) rectangles of unit width.
   (a) By considering the sum of the areas of these rectangles, show that:
   \[ \sum_{k=1}^{n} \frac{1}{k^2} < \frac{2n-1}{2n} \]  
   \[ \text{-- [5]} \]
   (b) Use a similar method to find, in terms of \( n \), a lower bound for \( \sum_{k=1}^{n} \frac{1}{k^2} \). [6]

3. The curve \( C \) has parametric equations:
   \( x = e^{t} - 4t + 3, \ y = 8 e^{\frac{1}{2}t}, \) for \( 0 \leq t \leq 2 \)
   (a) Find, in terms of \( \theta \), the length of \( C \). [5]
   (b) Find, in terms of \( \pi \) and \( e \), the area of the surface generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis. [5]

4. The diagram shows the curve with equation \( y = x^2 \) for \( 0 \leq x \leq 1 \), together with a set of \( n \) rectangles of width \( \frac{1}{n} \).
   (a) By considering the sum of the areas of these rectangles, show that:
   \[ \int_0^1 x^2 \, dx < \frac{2n+3x+1}{6n^2} \]  
   \[ \text{-- [4]} \]
   (b) Use a similar method to find, in terms of \( n \), a lower bound for \( \int_0^1 x^2 \, dx \). [5]
5. The integral $I_n$, where $n$ is an integer, is defined by,

$\quad I_n = \int \frac{1}{2} (1-x^2)^{n/2} \, dx$

(a) Find the exact value of $I_0$. --[3]
(b) By considering $\frac{d}{dn} (x(1-x^2)^{-1/2})$, or otherwise, show that

$\quad n \cdot I_{n+2} = 2^{n-1} \cdot 3^{1/2} \cdot (n-1) \cdot I_n$ --[5]

(c) Find the exact value of $I_5$ giving your answer in the form $k \cdot \sqrt{3}$ where $k$ is a rational number to be determined. --[13]

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6. Let $I_n = \int_0^1 (1+3x)^n e^{-3x} \, dx$, where $n$ is an integer.

(a) Show that $3I_n = 1 - 4ne^{-3} + 3nI_{n-1}$ --[3]
(b) Find the exact value of $I_2$. --[3]

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7. The diagram shows the curve with equation $y = \ln x$ for $x \geq 1$, together with a set of $(N-1)$ rectangles of unit width.

(a) By considering the sum of areas of these rectangles, show that $\ln N! > N \ln N - N + 1$. --[5]
(b) Use a similar method to find, in terms of $N$, an upper bound for $\ln N!$. --[3]

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8. The curve $C$ has parametric equations,

$\quad x = \frac{1}{2} t^2 - \ln t, \quad y = 2t + 1, \quad \text{for} \quad \frac{1}{2} \leq t \leq 2$

(a) Find the exact length of $C$. --[5]
(b) Find $\frac{dy}{dx}$ in terms of $t$, simplify your answer. --[4]

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9. It is given that, for \( n > 0 \), \( I_n = \int_0^1 x^n e^{x^2} \, dx \)

(i) Show that \( I_1 = \frac{1}{3} (e - 1) \) \( \quad [3] \)

(ii) Show that, for \( n \geq 3 \), \( 3I_n = e - (n - 2)I_{n-2} \) \( \quad [3] \)

(iii) Hence find the exact value of \( I_8 \) \( \quad [S.19/11/Q4] \)

10. A curve \( C \) is defined parametrically by \( x = \frac{e^t - e^{-t}}{e^t + e^{-t}} \) and \( y = \frac{e^t - e^{-t}}{e^t + e^{-t}} \), for \( 0 \leq t \leq 1 \). The area of the surface generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis is denoted by \( S \).

(i) Show that \( S = 4\pi \int_0^1 \frac{e^{-t} - e^t}{(e^t + e^{-t})^2} \, dt \) \( \quad [5] \)

(ii) Using the substitution \( u = e^t + e^{-t} \), or otherwise, find \( S \) in terms of \( \pi \) and \( e \) \( \quad [S.19/11/Q5] \)

11. Let \( I_n = \int_0^\frac{1}{\sqrt{\pi}} \cot^n x \, dx \), where \( n \geq 0 \)

(i) By considering \( \frac{d}{dx} (\cot^{n+1} x) \), or otherwise, show that \( I_{n+2} = \frac{1}{n+1} - I_n \) \( \quad [5] \)

The curve \( C \) has equation \( y = \cot x \), for \( \frac{1}{2}\pi < x < \frac{1}{2}\pi \)

(ii) Find, in exact form, the \( y \)-coordinate of the centroid of the region enclosed by \( C \), the line \( x = \frac{1}{4}\pi \) and the \( x \)-axis \( \quad [S.19/13/Q10] \)

12. The integral \( I_n \), where \( n \) is a positive integer, is defined by \( I_n = \int_0^\frac{1}{\sqrt{\pi}} \sin^n x \, dx \)

(i) Show that \( n + 1) I_{n+2} = 2^n n + \pi - \pi^2 I_n \) \( \quad [5] \)

(ii) Find \( I_5 \) in terms of \( \pi \) and \( I_1 \) \( \quad [S.19/11/Q3] \)
13. The curve $C$ is defined parametrically by:
\[ x = e^{-t} \cos t, \quad y = \frac{1}{4} e^t \]
Find the length of the arc of $C$ from the point where $t = 0$ to the point where $t = 3$.

14. (i) Using the substitution $u = \tan x$, or otherwise, find:
\[ \int \sec^2 x \tan^2 x \, dx \]

It is given that, for $n > 0$, $I_n = \int \sec^n x \, dx$.

(ii) Using the result that $\frac{d}{dx} (\sec^n x) = n \sec^{n-2} x \tan x$, show that, for $n \geq 2$,
\[ (n+1) I_n = (\sqrt{2})^{n-2} I_{n-2} \]

(iii) Hence find the mean value of $\sec^4 x \tan^2 x$ with respect to $x$, over the interval $0 < x < \frac{\pi}{2}$, giving your answer in exact form.

15. (i) Show that:
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x \, dx = \frac{1}{2} (e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}) \]

(ii) It is given that, for $n > 0$, $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos^n x \, dx$.

Show that, for $n \geq 2$,
\[ 4I_n = n(n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos^{n-2} x \cos^2 x \, dx \]

and deduce the reduction formula,
\[ (n^2 + 4) I_n = n(n-1) I_{n-2} \]

(iii) Using the result in part (i) and the reduction formula in part (ii), find the $y$-coordinate of the centroid of the region bounded by the $x$-axis and the arc of the curve $y = e^x \cos x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

Give your answer correct to 3 s.f.
16. A curve is defined parametrically by 
\[ x = t - \frac{1}{2} \sin^2 t \quad \text{and} \quad y = \sin^2 t \]
The arc of the curve joining the point where \( t = 0 \) to the point where \( t = \pi \) is rotated through one complete revolution about the \( x \)-axis. The area of the surface generated is denoted by \( S \).

(i) Show that \[ S = a \pi \int_0^\pi \sin^3 t \, dt \]
where the constant \( a \) is to be found. \[ -[5] \]
(ii) Using the result \( \sin 3t = 3 \sin t - 4 \sin^3 t \), find the exact value of \( S \).

\[ W-18/11/Q4 ] \[ -[3] \]

17. The curve \( C \) has equation \( x^2 + 2xy = y^3 - 2 \)

(i) Show that \( A(-1, 1) \) is the only point on \( C \) with \( x \)-coordinate equal to \(-1\). \[ -[2] \]

for \( n \geq 1 \), let \( A_n \) denote the value of \( \frac{d^n y}{dx^n} \) at the point \( A(-1, 1) \)

(ii) Show that \( A_1 = 0 \) \[ -[3] \]
(iii) Show that \( A_2 = \frac{3}{2} \)

\[ \text{let} \quad I_n = \int_{-1}^0 x^n \frac{d^ny}{dx^n} \, dx \]

(iv) Show that for \( n \geq 2 \), \[ I_n = (-1)^{n+1} \frac{3n+1}{3n+2} A_n - 3n I_{n-1} \]

(v) Deduce the value of \( I_3 \) in terms of \( I_1 \). \[ W-18/11/Q1(ii) \]

18. Let \( I_n = \int_{-1}^1 (x^2 - 1)^n \, dx \)

(i) Show that, for \( n \geq 1 \), \[ (2n+1)I_n = \frac{1}{2} - 2n I_{n-1} \]

(ii) Using the substitution \( x = \sec \theta \), show that \[ I_n = \int_0^\frac{\pi}{2} \tan^{2n+1} \theta \sec \theta \, d\theta \]

(iii) Deduce the exact value of \( I_3 \) (where \( I_3 = \int_0^\frac{\pi}{2} \sin^3 \theta \, d\theta \)). \[ W-18/12/11(iii) \]
19. Let \( I_n = \int_0^{\frac{1}{2} \pi} x^n \sin x \, dx \).

(i) Prove that, for \( n \geq 3 \), \( I_n + n(n-1)I_{n+1} = n(2\pi)^{n-1} \).

(ii) Calculate the exact value of \( I_1 \), and deduce the exact value of \( I_3 \). [5-17/11 Q 6] -- [3]

20. The curve \( C \) has polar equation \( \rho = a(1 + \sin \theta) \) for \( -\pi < \theta < \pi \), where \( a \) is a positive constant.

(i) Sketch \( C \). -- [2]

(ii) Find the area of the region enclosed by \( C \). -- [4]

(iii) Show that the length of the arc of \( C \) from the pole to the point furthest from the pole is given by:

\[
\int_{-\pi}^{\pi} \sqrt{1 + 4 \sin^2 \theta} \, d\theta
\]

(iv) Show that the substitution \( u = 1 + \sin \theta \) reduces the integral for \( s \) to:

\[
\int_0^1 \frac{1}{\sqrt{2-u}} \, du
\]


21. The curve \( C \) has equation \( y = \frac{1}{2} (e^x + e^{-x}) \) for \( 0 \leq x \leq 4 \).

(i) The region \( R \) is bounded by \( C \), the \( x \)-axis, the \( y \)-axis and the line \( x = 4 \). Find, in terms of \( e \), the coordinates of the centroid of the region \( R \). -- [4]

(ii) Show that \( ds = \sqrt{1 + e^{2x}} \), where \( s \) denotes the arc length of \( C \). Find and find surface area generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis. -- [4]

[5-17/11 Q 12]

22. A curve \( C \) has parametric equations

\[
x = \frac{2}{5} t^{\frac{3}{2}} - \frac{2}{3} t^{\frac{5}{2}}, \quad y = \frac{4}{3} t^{\frac{3}{2}}, \quad \text{for } 1 \leq t \leq 4.
\]

(i) Find the exact value of the arc length of \( C \). -- [5]

(ii) Find also the exact value of the surface area generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis. -- [3]

[5-17/13 Q 5]
2.3 Let $I_n$ denote $\int_0^{\pi/2} (4+2^2)^m dx$

(i) Find $\frac{d}{dx} (x(4+2^2)^n)$ and hence show that:

$$8nI_{n+1} = (3n-1)I_n + 2x^3 - n$$

(ii) Use the result for integrating $\frac{1}{x^2+a^2}$ with respect to $x$.

to find the value of $I_1$ and deduce that $I_3 = \frac{3\pi}{128} + \frac{1}{128}$

[5.17/13/26]

2.4 A curve $C$ has polar equation $r = 2a \cos (2\theta + \pi/4)$ for $0 < \theta < 2\pi$,

where $a$ is a positive constant.

(i) Show that $r = -2a \sin 2\theta$ and sketch $C$. [4]

(ii) Deduce that the Cartesian equation of $C$ is:

$$(x^2+y^2)^{3/2} = -4axy$$

[3]

(iii) Find the area of one loop of $C$.

[5]

(iv) Show that at the points (other than the pole) at which a tangent to $C$ is parallel to the initial line,

$$2\tan \theta = -5a2\theta$$

[3]

[5.17/13/11]

2.5 Let $I_n = \int_0^{\pi/2} \sec^n x \, dx$ for $n > 0$

(i) Find the value of $I_2$. [2]

(ii) Show that, for $n > 2$:

$$I_{n-1} = \frac{1}{n-1} I_{n-2}$$

[5]

(iii) The curve $C$ has equation $y = \sec^3 x$ for $0 < x < \frac{1}{2} \pi$.

The region $R$ bounded by $C$, the $x$-axis, the $y$-axis and the line $x = \frac{1}{2} \pi$. Find the volume of revolution generated when $R$ is rotated through $2\pi$ radians about the $x$-axis.

[4]

[5.17/11/08]

2.6 The polar equation of a curve $C$ is $r = a/(1+\cos \theta)$ for $0 < \theta < 2\pi$,

where $a$ is a positive constant.

(i) Sketch $C$. (continued) [3]
26(ii) Show that the cartesian equation of \( C \) is
\[ x^2 + y^2 = a\left( x + \sqrt{x^2 + y^2} \right) \]
-- (2)
(iii) Find the area of the sector of \( C \) between \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \).
(iv) Find the arc length of \( C \) between the point, \( \theta = 0 \) and the point where \( \theta = \frac{\pi}{2} \).

27. A curve \( C \) has polar equation \( r^2 = 8 \cos 2\theta \) for \( 0 < \theta < \frac{\pi}{2} \).
(i) Find a cartesian equation of \( C \).
-- (3)
(ii) Sketch \( C \).
-- (2)
(iii) Determine the exact area of the sector bounded by the arc of \( C \) between \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{\pi}{2} \), the half-line \( \theta = \frac{\pi}{4} \) and the half-line \( \theta = \frac{\pi}{2} \).

\[ \text{[It is given that } \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{\cos x} \, dx = \ln \left( \tan \frac{\pi}{4} \right) + c \]  
-- (3)

28. Let \( I_n = \int_0^{\frac{\pi}{2}} \tan^n x \, dx \), for \( n \geq 0 \), by differentiating
\( (n-1)I_{n-2} \) with respect to \( n \), prove that
(i) \( (n+2)I_n = (n-1)I_{n-2} \) for \( n \geq 2 \)
-- (5)
(ii) Hence find the exact value of \( I_4 \).
-- (4)

29. A curve \( C \) has parametric equations
\[ x = e^t \cos t, \quad y = e^t \sin t, \quad \text{for } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \]
(i) Find the arc length of \( C \).
-- (5)
(ii) Find the area of the surface generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis.
-- (8)

30. (i) The curve has equation \( y = -\ln (1-x^2) \) for \( -\frac{1}{2} \leq x \leq \frac{1}{2} \).
Show that \( 1 + \left( \frac{dy}{dx} \right)^2 = \left( \frac{1 + x^2}{1 - x^2} \right)^2 \)
-- (2)
(ii) Show further that \( 1 + x^2 \) may be expressed in the form \( A^2 + \frac{9}{x^2} + R \), where \( A \), \( R \) and \( R \) are the constants to be determined.
-- (3)
(iii) Find the exact arc length of \( C \).
-- (4)
31. Let \( I_n = \int_0^1 x^n (4-x^2)^{3/2} \, dx \), for \( n \geq 1 \), by considering \( \frac{d}{dx} \left[ x^n (4-x^2)^{3/2} \right] \)

(i) Show that:
\[(n+3) I_{n+1} - 4 n I_{n-1} \text{ where } n \geq 2 \] \[5-16/13/10 \]

(ii) Find the value of \( I_3 \) and deduce the exact value of \( I_2 \) \[5-16/13/10 \]

32. A curve has polar equation \( r = \frac{1}{1 - \sin \theta} \), for \( 0 \leq \theta \leq 2\pi \).

(i) Find in the form \( y^2 = f(x) \) the cartesian equation of the curve. \[1-3 \]

(ii) Hence sketch the curve, and shade the region whose area is given by \( \frac{1}{2} \int_{\pi/2}^{3\pi/2} \frac{1}{(1 - \sin \theta)^2} \, d\theta \) \[5-16/13/10 \]

(iii) Using the cartesian equation of the curve, find the area of this region. \[5-16/13/10 \]

33(i) Given that \( y \) is a function of \( x \) and that \( x = e^u \), show that:
\[ \frac{dy}{dx} = \frac{dy}{du} \text{ and } x \frac{d^2y}{dx^2} = \frac{d^2y}{du^2} + \frac{dy}{du} \]

(ii) Given also that, \( x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 17y = 34 \ln x + 21 \),

\[ \frac{d^2y}{du^2} + 2 \frac{dy}{du} + 17y = 34u + 21 \] \[1-1 \]

(iii) Find in terms of \( x \) given that \( y = 0 \) and \( \frac{dy}{dx} = -1 \) when \( x = 1 \) \[5-16/13/10 \]

34(i) Evaluate \( \int_{\pi/2}^{3\pi/2} x \sin x \, dx \) \[5-16/13/10 \]

(ii) Given that \( I_n = \int_0^{\pi/2} x^n \sin x \, dx \), prove that, for \( n > 1 \),
\[ I_n = n \left( \frac{1}{2} \right)^{n-1} - n(n-1) I_{n-2} \] \[5-16/13/10 \]

(iii) By first using the substitution \( x = a \sin \theta \), find the value of
\[ \int_0^1 (a \sin \theta)^3 \, d\theta \]
giving your answer in an exact form. \[5-16/11/9 \]
3.5. A curve has parametric equations.
(a) \( x = 1 - 3t^2, \ y = t(1 - 3t^2), \) for \( 0 \leq t \leq \frac{1}{\sqrt{3}} \)

Show that \( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = (1 + 9t^2)^2 \)

-- [2]

(b) Hence find: (i) the arc length of \( C \)

(ii) the surface area generated when \( C \) is rotated through \( 2\pi \) radians about the \( x \)-axis.

-- [3]

(c) Use the fact that \( t = y/x \) to find a cartesian equation of \( C \).

Hence show that the polar equation of \( C \) is \( \rho = \sec \theta (1 - 3\sin^2 \theta) \),

and state the domain of \( \theta \).

Find the area of the region enclosed between \( C \) and the initial line.

[-- [w-16/11/Q 11] -- [3]]

3.6. The curves \( C_1 \) and \( C_2 \) have polar equations.

\( C_1: \ \rho = \frac{1}{2}, \) for \( 0 \leq \theta \leq 2\pi \)

\( C_2: \ \rho = \sqrt{\cos \theta}, \) for \( 0 \leq \theta \leq \pi \)

(i) Find the polar coordinates of the point of intersection of \( C_1 \) and \( C_2 \).

(ii) Sketch \( C_1 \) and \( C_2 \) on the same diagram.

-- [3]

(iii) Find the exact value of the area of the region enclosed by \( C_1, C_2 \) and the half-line \( \theta = 0 \)

-- [5]

37ii) Let \( I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx \), where \( n \) is a non-negative integer,

Show that: \( I_n = n(\frac{1}{2}\pi)^{n-1} - n(n-1)I_{n-2} \), for \( n \geq 2 \)

-- [5]

(ii) Find the exact value of \( I_4 \).

[-- [w-15/11/Q 7] -- [4]]

3.8. The curve \( C \) has parametric equations,

\( x = 4t + 2t^{3/2}, \ y = 4t - 2t^{3/2}, \) for \( 0 \leq t \leq 4 \)

(i) Find the arc length of \( C \), giving your answer correct to 3 s.f.

-- [6]

(ii) Find the mean value of \( y \) with respect to \( x \) over the interval \( 0 \leq x \leq 32 \).

-- [5]
39. The curve C has polar equation \( r = a^2 \cos^2 \theta \) for \( 0 \leq \theta \leq \alpha \), where \( \alpha \) is measured in radians. The length of C is 2015. Find the value of \( \alpha \). 

40. Let \( I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta \), where \( n \) is a non-negative integer.
   
   (i) Use the identity \( \sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} \) to show that \( I_n + I_{n-1} = \frac{2}{2n-1} \), for all positive integer \( n \). - [5]
   
   (ii) Find the exact value of \( I_{1/2} \). - [4]

41. (i) It is given that \( I_n = \int_0^1 (ln x)^n dx \), for \( n \geq 0 \), show that \( I_n = (n-1) I_{n-1} - I_{n-2} \), for \( n \geq 2 \). - [6]

   (ii) Hence find, in an exact form, the mean value of \( (ln x)^3 \) with respect to \( x \) over the interval \( 1 \leq x \leq e \). - [6]

42. (i) The curve C has polar equation, \( r = a (1 - \sin \theta) \) for \( 0 \leq \theta < 2\pi \), sketch C.

   (ii) Find the area of the region enclosed by the arc of C for which \( \frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \), the half line \( \theta = \frac{\pi}{2} \) and the half line \( \theta = \frac{\pi}{2} \). - [5]

   (iii) Show that \( \frac{ds}{d\theta} = 4a^2 \sin^2 \left( \frac{\theta}{2} \right) \), where \( s \) denotes arc length, and find the length of the arc of C for which \( \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \). - [7]
1. \[ \int_0^1 \frac{1}{\sqrt{(3+4x-x^2)}} \, dx = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-(x-\frac{1}{2})^2}} \, dx = \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \frac{1}{\sqrt{2}} \arcsin(x) \bigg|_0^1 = \frac{\pi}{4} \]

2. (a) Sum of areas of rectangles:
\[ S = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n}{n} = 1 \]

(b) Sum of areas of approximate rectangles:
\[ S = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n}{n} = 1 \]

Thus, \[ S = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n}{n} = 1 \]

3. (a) \[ \frac{dx}{dt} = e^{-t} - 4t + 3, \quad \frac{dy}{dt} = 8e^{\frac{1}{2}t} \]
\[ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \]
\[ = \sqrt{e^{2t} + 16 + 8e^t} = e^t + 4 \]

Thus, length \[ = \int_0^2 (e^{t} + 4) \, dt = \left[ e^t + 4t \right]_0^2 = e^2 + 7 \]

(b) \[ S = 2\pi \int_0^a \frac{8e^{Lt}}{(e^{Lt} + 4)} \, dt = 2\pi \int_0^a \frac{8e^{Lt}}{1 + 4e^{Lt}} \, dt = 2\pi \left[ \frac{3}{5} \int ds \right] = \frac{3}{5} \pi \left[ e^{2t} + 4t \right]_0^2 = \frac{3}{5} \pi (e^{2} + 8e - 36) \]

4. (a) \[ \int_0^1 x^2 \, dx = \frac{1}{3} \left[ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right] = \frac{1}{3} \left[ \frac{3}{2} \right] = \frac{1}{2} \]

(b) \[ \int_0^1 x^2 \, dx = \frac{1}{3} \left[ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right] = \frac{1}{3} \left[ \frac{3}{2} \right] = \frac{1}{2} \]

5. (a) \[ I_1 = \int_0^1 \frac{1}{1-x^2} \, dx = \int \frac{1}{1-x^2} \, dx = \tan^{-1}(x) \bigg|_0^1 = \frac{\pi}{4} \]

(b) \[ I_2 = \frac{\pi}{4} \]

(c) \[ I_3 = \frac{\pi}{4} \]

6. (a) \[ I_n = \int e^{-x} \sin(2x) \, dx \]

(b) \[ I_0 = \int e^{-x} \sin(2x) \, dx = -\frac{1}{3} e^{-3x} \sin(2x) \bigg|_0^1 = -\frac{1}{3} e^{-3} \sin(2) + \frac{1}{3} (1-e^{-3}) \]

\[ I_1 = \frac{1}{3} (1-4e^{-3}) \]

\[ I_2 = \frac{1}{3} (1-6e^{-3} + 2(2-5e^{-3}) \int_0^1 e^{-x} \sin(2x) \, dx = \frac{1}{3} (5 - 26e^{-3}) \]
7. (a) \[ \ln N! = \ln 2 + \ln 3 + \cdots + \ln N \]
\[ > \int_1^N \ln x \, dx \]
\[ = \left[ x \ln x \right]_{1}^{N} - \int_1^N dx \]
\[ = N \ln N - N + 1 \]
\[ = N \ln N - N + 1 \]
\[ \therefore \ln N! > N \ln N - N + 1 \]

(b) \[ \ln 1 + \ln 2 + \cdots + \ln N - 1 \]
\[ < \int_1^N \ln x \, dx \ (\text{or} < \int_1^N x \, dx) \]
\[ \Rightarrow \ln N! < N \ln N - N + 1 + N - N^2 \]
\[ \Rightarrow \ln N! < (N + 1) \ln N - N + 2 - (N - 1)^2 \]

8. (a) \[ x = \frac{1}{2} t^2 - \ln t, \quad y = 2t + 1 \]
\[ \frac{dy}{dt} = \frac{t - t^{-1}}, \quad \frac{dx}{dt} = 2 \]
\[ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = t^2 - 2 + t^{-2} + 4 \]
\[ = \frac{15}{2} + 2 \ln 2 \]

(b) \[ \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{2}{t - t^{-1}} \]
\[ \frac{d}{dt} (\frac{dy}{dx}) = -2 (t - t^{-2}) \]
\[ \left( \frac{dy}{dx} \right)^2 = 4 \left( t - t^{-2} \right)^2 \]
\[ \left( \frac{d}{dt} (\frac{dy}{dx}) \right)^2 = \frac{15}{2} \]

9. (i) \[ I_3 = \int_0^1 x^2 e^x \, dx = \frac{1}{3} (e^1 - 1) \]

(ii) \[ I_3 = \int_0^1 x^2 e^x \, dx \]
\[ = \frac{1}{3} \left[ x^2 e^x \right]_0^1 - \frac{2}{3} \int_0^1 x e^x \, dx \]
\[ = e - \frac{2}{3} \int_0^1 x e^x \, dx \]
\[ \Rightarrow I_3 = \int_0^1 x e^x \, dx = 3 \]

(iii) \[ I_5 = \int_0^1 x^2 e^x \, dx = \frac{1}{3} (e - e + 1) = \frac{1}{3} \]

10. (a) \[ \frac{d}{dt} (e^t + e^{-t}) = \frac{2}{t - t^{-1}} \]
\[ \frac{dy}{dt} = (e^t + e^{-t}) - \frac{2}{t - t^{-1}} \]
\[ \frac{d}{dt} (\frac{dy}{dt}) = \frac{4}{(e^t + e^{-t})^2} \]
\[ S = 2 \pi \int_0^1 \frac{dy}{dt} \, dt \]
\[ = 2 \pi \int_0^1 \frac{e^t + e^{-t}}{(e^t + e^{-t})^2} \, dt \]
\[ = 4 \pi \int_0^1 t e^t \, dt \]
\[ = 4 \pi \left[ \frac{1}{2} \frac{e^t}{e^t} \right]_0^1 \]
\[ = 4 \pi \left( \frac{1}{2} - \frac{1}{e + 1} \right) \]
11 (i) \[ \frac{d}{dx} \left( \text{Get}^{n+1} \right) = -(n+1) \text{Get}^n \text{ cos}^2 \theta \]

\[ = -(n+1) \text{Get}^n \left( \text{Get}^{n+1} \right) \]

\[ = -(n+1) \text{Get}^{n+1} - (n+1) \text{Get}^n \]

\[ \Rightarrow 0 = -(n+1) \left( \text{I}_{n+2} + \text{I}_n \right) \]

\[ \Rightarrow \text{I}_{n+2} = -\frac{1}{n+1} - \text{I}_n \]

(ii) \[ \int \text{Get}^2 \, dx = \text{I}_2 + \frac{2}{3} \text{A} \]

\[ A = \int \text{Get} \, dx = \left[ \ln \text{Get} \right]_0^1 \]

\[ I_2 = 1 - I_0 = 1 - \frac{1}{2} \pi \]

\[ I = \frac{1}{I_2} \left( 1 - \frac{1}{2} \pi \right) \]

12 (i) \[ I_{n+2} = \left[ \frac{x^{n+1}}{n+1} \right] - x^n \left[ \frac{x^{n-1}}{n-1} \right] \]

\[ = \frac{2^{n+1}}{n+1} + \frac{n+1}{n+1} \left( \frac{x^n}{n} - \frac{x^{n+1}}{n} \right) \]

\[ \Rightarrow (n+1) \text{I}_{n+2} = 2^{n+1} + n \left( \frac{x^n}{n} - \frac{x^{n+1}}{n} \right) \]

\[ \Rightarrow \frac{\text{I}_{n+2}}{\text{I}_n} = 2 \left( \frac{x^n}{n} - \frac{x^{n+1}}{n} \right) \]

(iii) \[ I_2 = 1 + \pi - \pi^2 \]

\[ I_5 = 4 + \pi - \pi^2 \]

15 (i) \[ \int \left[ \frac{2}{3} e^x \, \cos x \right] \, dx \]

\[ I = \int \frac{2}{3} e^x \, \cos x \, dx \]

\[ = \left[ e^x \sin x \right]_0^1 - \int_{0}^{1} e^x \sin x \, dx \]

\[ = \left[ e^x \sin x \right]_0^1 - I \]

\[ 2I = e^1 + e^{-1} \]

\[ I = \frac{1}{2} \left( e^1 + e^{-1} \right) \]

13. \[ \frac{dy}{dt} = e^t-1 \quad \text{and} \quad \frac{dy}{dt} = 2e^t \]

\[ \left( \frac{dy}{dt} \right)^2 = \left( \frac{dy}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = (e^t + 1)^2 \]

\[ \text{arc length} = \int_{0}^{1} \left( e^t + e^{-t} \right) \, dt = \frac{e^{1} + e^{-1}}{2} \]
15(ii) \[ I_n = \left[ \frac{\pi}{2} e^{\pi x} \right]^{\pi}_{-\pi} + \int_{-\pi}^{\pi} e^{\pi x} e^{-|x|} dx \]

\[ = 0 - \frac{\pi}{4} \int_{-\pi}^{\pi} e^{\pi x} (e^{2\pi} x + (n+1)e^{\pi x} e^{-\pi x}) dx \]

\[ \Rightarrow 4I_n = n(n-1) \int_{-\pi}^{\pi} e^{\pi x} e^{-|x|} dx - nI_n \]

\[ \Rightarrow (n+4)I_n = n(n-1) \int_{-\pi}^{\pi} e^{\pi x} (1 - e^{2\pi} x) e^{-|x|} dx \]

\[ (n+4)I_n = n(n-1)I_{n-2} - n(n-1)I_n \]

\[ \Rightarrow (n^2+4)I_n = n(n-1)I_{n-2} \checkmark \]

16(i) \[ \frac{dx}{dt} = 1 - \cos t = 2 \sin^2 t, \quad \frac{dy}{dt} = 2 \sin t \cos t \]

\[ \frac{ds}{dt} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = \sqrt{4 \sin^4 t + 4 \sin^2 t \cos^2 t} = 2 \sin t \]

\[ S = 2\pi \int_{0}^{\pi} y \frac{ds}{dt} dt = 2\pi \int_{0}^{\pi} \sin t \cdot 2 \sin t dt = 2 \pi \]

\[ S = 4\pi \int_{0}^{\pi} \sin^3 t dt \Rightarrow q = 4\pi \]

\[ \int_{-\pi}^{\pi} x \frac{dy}{dx} dy = \int_{-\pi}^{\pi} x \frac{dy}{dx} dx \]

\[ \Rightarrow \frac{dy}{dx} = \frac{\pi}{4} \int_{-\pi}^{\pi} x \frac{dy}{dx} dx \]

\[ \Rightarrow A_2 = 3[A_1 - 2I_2] \]

\[ A_2 = \frac{3}{5} + 6I_2 \]

\[ \checkmark \]
18. (i) \[ I_n = \left[ \pi (x^2-1)^{\frac{\pi}{2}} \right] = 2n \int x^2 (x^2-1)^{n-1} \, dx \]
\[ = \frac{\pi}{2} - 2n \int x^2 (x^2+1)(x^2-1)^{n-1} \, dx \]
\[ I_n = \frac{\pi}{2} - 2n I_n - 2n I_{n-1} \]
\[ \Rightarrow (2n+1) I_n = \frac{\pi}{2} - 2n I_{n-1} \]

(ii) \[ \text{But } x = \cos \theta \Rightarrow \frac{dx}{d\theta} = -\sin \theta \]
\[ \text{Then } \theta = \frac{\pi}{2} \Rightarrow \theta = 0 \text{ at } x = 1 \Rightarrow \theta = 0 \]
\[ I_n = \int_0^\pi (\cos^2 \theta - 1) \cos x \cos \theta \, d\theta \]
\[ = \int_0^\pi \cos^2 \theta \, d\theta \]

(iii) \[ s = \int_0^\pi \cos^2 \theta \, d\theta \]

(iv) \[ u = 1 + 2 \sin \theta \Rightarrow \frac{du}{d\theta} = 2 \cos \theta \]
\[ s = \int_0^\pi (1 + 2 \sin \theta + 2 \cos \theta)^2 \, d\theta \]

19. (i) \[ I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx \]
\[ = \left[ -x^n \cos x \right]_0^{\frac{\pi}{2}} + \frac{n}{2} \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx \]
\[ = 0 + \frac{n}{2} (\sin x - x \cos x) \]
\[ I_n = n \left( \frac{\pi}{2} \right)^{n-1} - n (n-1) I_{n-2} \]
\[ \Rightarrow I_n + n(n-1) I_{n-2} = n \left( \frac{\pi}{2} \right)^{n-1} \]

(ii) \[ I_1 = \int_0^{\frac{\pi}{2}} x \sin x \, dx = \left[ -x \cos x \right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin x \, dx \]
\[ = 1 \]

\[ n = 3 \Rightarrow I_3 = 3 \left( \frac{\pi}{2} \right)^2 - 3x^2 \times 1 = \frac{\pi^3}{2} - 6 \]
Integration

22. (i) \[ \frac{dx}{dt} + \frac{dy}{dt} = \left( t^{3/2} - t^{-1/2} \right) + \sqrt{t} \]
\[ = \left( t^{3/2} + t^{-1/2} \right)^2 \]
\[ S = \int_1^4 \left( t^{3/2} + t^{-1/2} \right) dt \]
\[ = \left[ \frac{2}{3} t^{3/2} + \frac{2}{1} t^{1/2} \right]_1^4 \]
\[ = \left( \frac{2}{3} \cdot 4^{3/2} + \frac{2}{1} \cdot 4^{1/2} \right) - \left( \frac{2}{3} \cdot 1^{3/2} + \frac{2}{1} \cdot 1^{1/2} \right) \]
\[ = \frac{72}{3} \]

(ii) \[ S = \int_1^4 \sqrt{y} \cdot \frac{dy}{dt} \ dt \]
\[ = 2\pi \int_3^4 \frac{4}{3} t^{3/2} (t^{3/2} - t^{-1/2}) \ dt \]
\[ = \frac{8}{3} \pi \int_3^4 \left( \frac{4}{3} t^{3/2} + \frac{2}{3} t^{-1/2} \right) \]
\[ = \frac{8}{3} \pi \left( \frac{4}{3} t^{3/2} - \frac{2}{3} t^{-1/2} \right)_3^4 \]
\[ = 190 \pi \]

23. (i) \[ I_n = \int_0^1 \frac{dx}{(x + x^2)^n} \]
\[ = \int_0^1 \frac{dx}{(x + x^2)^n} \]
\[ = \int_0^1 \frac{dx}{2^n (x + x^2)^{n-1}} \]
\[ \Rightarrow 2^{-n} = I_n - 2n I_{n+1} + 8n I_{n+2} \]

Now \[ I_2 = \int_0^1 \frac{dx}{(x + x^2)^2} \]
\[ I_3 = \int_0^1 \frac{dx}{(x + x^2)^3} \]
\[ \Rightarrow I_3 = 3\pi + \frac{1}{38} \]

Answers

24. (i) \[ 24, 2 = 2a \cos (18 + \frac{3\pi}{2}) \text{ for } 0 \leq \theta \leq 3\pi \]
\[ (ii) 2 = 2a \cos \theta \]
\[ \Rightarrow -2a \sin \theta \]

(iii) \[ A = -4a \sin \theta \cos \theta \]
\[ \Rightarrow 4 \sin \theta \]

(iii) Area of one loop
\[ = \frac{1}{2} \int_{-\alpha}^{\alpha} 2a^2 \sin^2 \theta \ d\theta \]
\[ = a^2 \int_{-\alpha}^{\alpha} \sin^2 \theta \ d\theta \]
\[ = a^2 \left[ \theta - \frac{1}{2} \sin 2\theta \right]_{-\alpha}^{\alpha} \]
\[ = a^2 \left[ \frac{\alpha}{2} \right] \]

(iv) \[ y = \alpha \sin \theta = -2a \cos \theta \]
\[ \frac{dy}{d\theta} = -2a \sin \theta \]
\[ \Rightarrow \frac{dy}{d\theta} = -2a \sin \theta \]

25. (i) \[ I_n = \int_0^{2a} x^n \ dx \]
\[ n > 0 \]

Now \[ I_2 = \int_0^{2a} x^2 \ dx \]
\[ I_3 = \int_0^{2a} x^3 \ dx \]
\[ \Rightarrow I_3 = \frac{3\pi}{32} + \frac{1}{38} \]

(continued →)
Volume of revolution = \( \pi \int_a^b \left( \frac{3}{2} x \right)^2 dx \)

\[ = \frac{9}{4} \pi \int_0^1 (1 + 3x) dx \]

(i) \( I_i = \int_0^1 (1 + 3x) dx = 4/3 \)

\[ \Rightarrow \frac{9}{4} \pi \left( \frac{4}{3} \right) = 9\pi \]

\[ \Rightarrow \quad \text{Volume of revolution} = \frac{28\pi}{15} \]

(ii) \( I_6 = \int_0^1 (1 + 3x)^2 dx = 28/15 \)

\[ \Rightarrow 2 \left( \frac{28\pi}{15} - \frac{28}{15} \right) = 2 \ln 3 \]

\[ \Rightarrow \quad \int_0^1 \left( 1 + 3x \right)^2 dx = \ln 9 \sqrt{e} \]

Now, \( I_1 = \int_0^1 3x dx = \frac{9}{2} \)

\[ \Rightarrow \int_0^1 \left( 1 + 3x \right)^2 dx = \frac{9}{2} + 3 \int_0^1 x dx \]

\[ \Rightarrow 0 = (n-1)I_{n-2} + (n-1)I_n + 3I_n \]

\[ \Rightarrow (n+2)I_n = (n-1)I_{n-2} \]

(iii) Surface area = \( \frac{9}{2} \int_0^1 (1 + 3x + 3\sqrt{2}x) dx = \frac{9}{2} \left[ \frac{3}{2} + 3\sqrt{2} \right] \]

\[ = \frac{9}{2} \left[ \frac{3}{2} + 3\sqrt{2} \right] \]

\[ = \frac{9}{2} \left[ 4\sqrt{2} + 9\sqrt{2} \right] \]

\[ = \frac{9}{16} \left[ 4\sqrt{2} + 9\sqrt{2} \right] \]

(iv) Length = \( \int_0^1 \sqrt{\left( 3 + 2(3x + \sqrt{3}x) \right)^2 + 1} dx = \frac{9}{2} \left[ 4\sqrt{2} + 9\sqrt{2} \right] \)

\[ = \frac{9}{2} \left( 4\sqrt{2} + 9\sqrt{2} \right) \]

\[ = \frac{9}{2} \left( 13\sqrt{2} \right) \]

\[ = 2a \sqrt{2} \]

\[ \Rightarrow \quad S = \frac{9}{2} \left( 13\sqrt{2} \right) \]

\[ = \frac{9}{2} \left( 13\sqrt{2} \right) \]

\[ = 21\sqrt{2} \text{ with } S > 32\sqrt{2} \]

(continued ->)
29(ii) $S = 2\pi \int_{\frac{\pi}{2}}^{\pi} y \frac{ds}{dt} \frac{dt}{dt} dt$

$= \int_{\frac{\pi}{2}}^{\pi} e^{4t} \cos t \sin t \cdot 2 \sqrt{2} \sin t \cos t \frac{dt}{dt} dt$

$= 4\sqrt{2} \pi \int_{\frac{\pi}{2}}^{\pi} e^{4t} \cdot 2 \sqrt{2} \sin t \cos t \frac{dt}{dt} dt$

$= 4\sqrt{2} \pi \int_{\frac{\pi}{2}}^{\pi} e^{4t} \frac{1}{2} \cdot 2 \sqrt{2} \sin t \cos t \frac{dt}{dt} dt$

$= 4\sqrt{2} \pi \int_{\frac{\pi}{2}}^{\pi} e^{4t} \cdot 2 \sqrt{2} \sin t \cos t \frac{dt}{dt} dt$

$= \left[ e^{4t} \right]_{\frac{\pi}{2}}^{\pi} = e^{4\pi} - e^{2\pi}$

Consider $I = \int_{\frac{\pi}{2}}^{\pi} e^{4t} \cos t \sin t \frac{dt}{dt} dt$

$= \left[ -\frac{e^{4t}}{4} \cos t \sin t \frac{dt}{dt} dt \right]_{\frac{\pi}{2}}^{\pi} - 4\int_{\frac{\pi}{2}}^{\pi} e^{4t} \cos t \sin t \frac{dt}{dt} dt$

$= \left( -\frac{e^{4\pi}}{4} \cos \pi \sin \pi - \left( -\frac{e^{2\pi}}{4} \cos \frac{\pi}{2} \sin \frac{\pi}{2} \right) \right) - 4\int_{\frac{\pi}{2}}^{\pi} e^{4t} \cos t \sin t \frac{dt}{dt} dt$

$= e^{2\pi} - 4\int_{\frac{\pi}{2}}^{\pi} e^{4t} \cos t \sin t \frac{dt}{dt} dt$

$= e^{2\pi} - 4I$

$\Rightarrow I = \frac{e^{2\pi}}{5}$

$S = 4\sqrt{2} \pi \left( \frac{e^{2\pi} - e^{4\pi}}{10} \right) = 2\sqrt{2} \pi \left( e^{2\pi} - e^{4\pi} \right)$

$\Rightarrow S = 2\sqrt{2} \pi \left( e^{2\pi} - e^{4\pi} \right)$

$30(i) \frac{dy}{dx} = \frac{2x}{1-x^2}$

$1 + \left( \frac{dy}{dx} \right)^2 = \frac{1 + 4x^2}{1-x^2} = \left( 1-x^2 \right)^2$

$= \left( 1 + x^2 \right)^2$

$= \left( 1 + x^2 \right)^2$

$= \left( 1 + x^2 \right)^2$

$S = \frac{1}{2} \int_{-1}^{1} \left( 1 + 2x \right) \frac{1}{2} dx$

$= \left[ \frac{1}{2} \left( 1 + 2x \right)^{\frac{3}{2}} \right]_{-1}^{1}$

$= \left[ \frac{1}{2} \left( 3 \right)^{\frac{3}{2}} - \frac{1}{2} \right]$

$= \frac{3}{2}$
33. \( x = e^u \) \( \& \ y = f(u) \)

\[ \text{Answers:} \quad C_1, C_2 = \frac{1}{2}, \quad 0 \leq \theta \leq 2\pi \]

(i) \( \frac{\partial y}{\partial u} = \frac{x \frac{dy}{du} - y \frac{dx}{du}}{x \frac{dx}{du}} \) for interaction of \( a \) and \( b \)

(ii) \( \frac{d^2 y}{dx^2} = \frac{x^2 \frac{d}{du} \left[ \frac{dy}{dx} \right]}{x^2 \frac{1}{dx} \left[ \frac{d^2 y}{dx^2} \right]} \)

\[ = -\frac{dy}{dx} + x \frac{d^2 y}{du^2} \]

Substituting in the diff. equn.

\[ \frac{d^2 y}{dx^2} = \frac{dy}{dx} + 3 \frac{dy}{dx} + 17y = 34u + 21 \]

\[ \Rightarrow \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 17y = 34u + 21 \]

(iii) \( m^2 + 2m + 17 = 0 \Rightarrow m = -1 \pm i \ln(2) \)

\[ y_c = e^{-u} \left( A \cos(4u) + B \sin(4u) \right) \]

\[ y_p = Cu + d \quad \Rightarrow y'' = 0 \]

\[ \Rightarrow 2C + 17Cu + 17d = 34u + 21 \]

\[ \Rightarrow C = 2, \quad d = 1 \Rightarrow y_p = 2u + 1 \]

\[ y = \frac{1}{2} \left[ A \cos(4u) + B \sin(4u) \right] + 2u + 1 \]

\[ y = 0, \quad x = 1 \Rightarrow A + 0 = 0 \Rightarrow x = 0, \quad y = 0 \]

\[ \frac{dy}{du} = \frac{1}{2} \left[ -A \sin(4u) + B \cos(4u) \right] - 2 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{34 \cos(4u) + 21}{2} \]

\[ y = 2 \ln x + \frac{1}{2} \left[ A \cos(4u) + B \sin(4u) \right] \]

\[ y = \frac{1}{2} \left( 1 - 3t^2 \right), \quad y = t \left( 1 - 3t^2 \right), \quad 0 \leq t \leq 1 \]

34.

(a) \[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = 36t^2 + 18t + \frac{81t^4}{4} \]

\[ = 1 + 18t^2 + \frac{81t^4}{4} \]

(b)(i) \[ S = \int_0^3 \left( 1 + 9t^2 \right) dt = \left[ t + 3t^3 \right]_0^3 = \frac{36}{2} \]

\[ = \frac{18}{2} \]

(b)(ii) \[ S = \int_0^3 \left( 1 + 9t^2 \right) dt = \int_0^3 \left( \frac{1}{2} t^2 + \frac{3}{2} t^2 + t \right) dt = \left[ \frac{1}{2} t^3 + \frac{3}{2} t^2 + \frac{1}{2} t \right]_0^3 = \frac{18}{2} \]

(continued)
Integration

37. (i) \[ I_n = \int_0^\frac{\pi}{2} x^n \sin^n x \, dx, \quad n > 0 \]
\[ = \left[ -x^n \cos x \right]_0^\frac{\pi}{2} + \int_0^\frac{\pi}{2} nx^{n-1} \sin^{n-1} x \cos x \, dx \]
\[ = 0 + \left[ nx^{n-1} \sin^2 x \right]_0^\frac{\pi}{2} - \frac{1}{2} \int_0^\frac{\pi}{2} n(n-1)x^{n-2} \cos^2 x \, dx \]
\[ = \frac{\pi}{2} \left[ 2(n-1) \sin^2 x \cos x \right]_0^\frac{\pi}{2} - \frac{n(n-1)}{2} \int_0^\frac{\pi}{2} \sin^2 x \, dx \]
\[ = \frac{\pi}{2} \cdot 2(n-1) \sin^2 x \left|_0^\frac{\pi}{2} \right. - \frac{n(n-1)}{2} \left[ \frac{x}{2} - \frac{\sin 2x}{4} \right]_0^\frac{\pi}{2} \]
\[ = \frac{\pi}{2} \cdot 2(n-1) - \frac{n(n-1)}{2} \left( \frac{\pi}{2} \right) \]
\[ = \pi(n-1) - n(n-1) \frac{\pi}{2} \]
\[ = \pi(n-1) - \frac{n(n-1)}{2} \]

(ii) \[ I_0 = \int_0^\frac{\pi}{2} \sin^2 x \, dx \]
\[ = \frac{\pi}{2} - \frac{1}{2} \]
\[ I_1 = \int_0^\frac{\pi}{2} \sin x \, dx \]
\[ = -\left[ \cos x \right]_0^\frac{\pi}{2} \]
\[ = -1 + 0 = -1 \]
\[ I_2 = \int_0^\frac{\pi}{2} \sin^2 2x \, dx \]
\[ = \int_0^\frac{\pi}{2} \left( 1 - \cos 4x \right) \, dx \]
\[ = \left[ x \right]_0^\frac{\pi}{2} - \frac{1}{4} \int_0^\frac{\pi}{2} \cos 4x \, dx \]
\[ = \frac{\pi}{2} - \frac{1}{4} \left[ \frac{\sin 4x}{4} \right]_0^\frac{\pi}{2} \]
\[ = \frac{\pi}{2} - 0 = \frac{\pi}{2} \]

38. (i) \[ \int_0^x t^2 \, dt \]
\[ = \left[ \frac{1}{3} t^3 \right]_0^x \]
\[ = \frac{1}{3} x^3 \]

(ii) \[ \int_0^x y \, dy \]
\[ = \left[ \frac{1}{2} y^2 \right]_0^x \]
\[ = \frac{1}{2} x^2 \]

\[ S = \int_0^\frac{\pi}{2} (3x + 18) \, dx \]
\[ = \left[ \frac{3}{2} x^2 + 18x \right]_0^\frac{\pi}{2} \]
\[ = \frac{3}{2} \left( \frac{\pi^2}{4} \right) + 18 \left( \frac{\pi}{2} \right) \]
\[ = \frac{3}{2} \pi^2 + 9 \pi \]

39. \[ e^x = \int_0^x \sqrt{1 + t^2} \, dt \]
\[ S = \int_0^\infty \sqrt{1 + t^2} \, dt \]
\[ = \left[ \frac{1}{2} \ln \left( t + \sqrt{1 + t^2} \right) \right]_0^\infty \]
\[ = \lim_{t \to \infty} \left( \frac{1}{2} \ln \left( t + \sqrt{1 + t^2} \right) \right) \]
\[ = \frac{1}{2} \ln \left( t + \sqrt{1 + t^2} \right) \]

\[ IV \]
\[ I_0 = \int_0^\frac{\pi}{2} \sin^2 x \, dx \]
\[ I_1 = \int_0^\frac{\pi}{2} \sin x \, dx \]
\[ I_2 = \int_0^\frac{\pi}{2} \sin^2 2x \, dx \]
\[ I_3 = \int_0^\frac{\pi}{2} 2 \, dx \]

\[ MV = \frac{I_3}{e-1} = 6 - 2e \]

\[ \Rightarrow e^x = \frac{105}{5} \Rightarrow x = \frac{105}{5} \]

\[ M.V. = S \]
\[ b-a \]

\[ \Rightarrow \frac{1}{16} e^{40} - \frac{1}{16} = 20/5 \]

\[ \Rightarrow e^{40} = 1955.837 \Rightarrow x = 1.89 \]
42. \( S = a \left( 1 - \cos \theta \right) \), \( 0 \leq \theta \leq \pi \\
\text{(ii) } S = 2 \times \frac{1}{2} a^2 \int_{\frac{\pi}{2}}^{\pi} \left( 1 - 2 \cos \theta + \cos^2 \theta \right) d\theta \\
= a^2 \int_{\frac{\pi}{2}}^{\pi} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \\
= a^2 \left[ \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{2}}^{\pi} \\
= a^2 \left( \frac{3}{4} \pi + 2 \right) \checkmark \\
\text{(iii) } \left( \frac{ds}{d\theta} \right)^2 = a^2 \left( 1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta \right) \\
= a^2 \left( 1 - \cos \theta \right) \\
= 2a^2 \left( 1 - \cos \theta \right) \\
= 4a^2 \sin^2 \frac{\theta}{2} \checkmark \\
\text{New arc length} \\
S = 2 \int_{\frac{\pi}{2}}^{\pi} 2a \sin \frac{\theta}{2} d\theta \\
= 4a \left[ -2 \cos \frac{\theta}{2} \right]_{\frac{\pi}{2}}^{\pi} \\
= 4 \sqrt{2} a \checkmark 

\text{X X X X X}