Pure Math 1

Functions
Revision

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Example 1: Functions $f$ and $g$ are defined by:

$f: x \rightarrow 3x+2, \quad x \in \mathbb{R}$

$g: x \rightarrow 4x-12, \quad x \in \mathbb{R}$

Solve the equation $f^{-1}(x) = gf(x)$. 

\[\text{Solution: } f(x) = 3x + 2 \quad \text{(1)}\]

or $y = 3x + 2 \quad \text{(2)}$

Interchange $x$ and $y$ in (2) to get $f^{-1}(x)$

\[x = 3y + 2\]

\[\Rightarrow y = \frac{x-2}{3}\]

or $f^{-1}(x) = \frac{x-2}{3} \quad \text{(3)}$

Now $gf(x) = g(f(x)) = g(3x+2)$

\[= 4(3x+2)-12 \quad \left(\text{Substituting } f(x) = 3x+2\right)\]

\[= 12x - 4 \quad \text{(4)}\]

Now to solve $f^{-1}(x) = gf(x)$

from (3) and (4)

\[\frac{x-2}{3} = 12x - 4\]

or $x - 2 = 36x - 12$

$35x = 10$

\[x = \frac{10}{35} = \frac{2}{7}\]

\[\therefore x = \frac{2}{7} \checkmark\]
Example 2: (a) The curve \( y = x^2 + 3x + 4 \) is translated by \((-2, 0)\). Find and simplify the equation of the translated curve.

(b) The graph of \( y = f(x) \) is transformed to the graph of \( y = 3f(-x) \). Describe fully the two single transformations which have been combined to give the resulting transformation.

Solution: (a) \( y = x^2 + 3x + 4 \) is translated by \((-2, 0)\).

\[ y = (x-2)^2 + 3(x-2) + 4 \]

or

\[ y = x^2 - x + 2 \]

(b) Now \( y = 3f(-x) \).

The two transformations are:

(i) Reflection in \( y \)-axis

(ii) Stretch factor 3 in \( y \)-direction.

Example 3: The graph of \( y = f(x) \) is transformed to the graph of \( y = 1 + f\left(\frac{x}{2}\right) \). Describe fully the two single transformations which have been combined to give the resulting transformation.

Solution: \( y = 1 + f\left(\frac{x}{2}\right) \)

Stretch, factor 2, \( x \)-direction (or \( y \)-axis invariant)

Translation or shift by 1 unit in \( y \)-direction.

or Translation/shift \((0, 1)\)
Example 4 (a) Express \(2x^2 + 12x + 11\) in the form \(a(x + a)^2 + b\), where \(a\) and \(b\) are constants.

The function \(f\) is defined by \(f(x) = 2x^2 + 12x + 11\) for \(x \leq -4\).

(b) Find an expression for \(f^{-1}(x)\) and state the domain of \(f^{-1}\).

The function \(g\) is defined by \(g(x) = 2x - 3\) for \(x \leq k\).

(c) For the case where \(k = -1\), solve the equation \(f(g(x)) = 193\).

(d) State the largest value of \(k\) possible for \(f(g)\) to be defined.

Solution (a) \(2x^2 + 12x + 11\)

\[= 2\left[x^2 + 6x + 9 - 9\right] + 11\]

\[= 2(x + 3)^2 - 18 + 11\]

\[= 2(x + 3)^2 - 7 \quad \text{for } 0\]

or \(g(x) = 2x^2 + 12x + 11\) for \(x \leq -4\)

but \(x = 2(y + 3)^2 - 7\) \(\text{or } 2(y + 3)^2 = x + 7\)

\[(y + 3)^2 = \frac{x + 7}{2}\]

\[y + 3 = \pm \sqrt{\frac{x + 7}{2}}\]

\[y = -3 \pm \sqrt{\frac{x + 7}{2}}\]

\[\text{Domain: for } x \leq -4\]

\[\text{Range: for } y \geq -5\]

\[f^{-1}(x) = -3 - \sqrt{\frac{x + 7}{2}}\]

\[\text{for } x = -4\]

\[\text{and } y = -5\]

\[\text{Range for } f^{-1}(x) \leq -4, x \geq -5\]

\[\text{fpw > } -5\]

(c) \(g(x) = 2x - 3\) for \(x \leq -1\)

\[f(g(x)) = f(2x - 3)\]

for \(0\)

\[= 2\left[2x - 3 + 3\right] - 7\]

\[= 8x^2 - 7\]

Now \(fg(x) = 8x^2 - 7 = 193\)

\[8x^2 = 200\]

\[x^2 = 25\]

\[x = \pm 5\]

\[x = -5 \text{ only}; x \leq -1\]

(d) \(f(g(x)) = f(2x - 3)\)

\[\text{Domain for } x \leq 0\]

\[\text{Range for } x \leq -\frac{1}{2}\]

\[\text{largest value of } k = -\frac{1}{2}\]
Example 5: Functions $f$ and $g$ are defined for $x \in \mathbb{R}$ by

\[ f: x \mapsto \frac{1}{2}x - a \]
\[ g: x \mapsto 3x + b \]

where $a$ and $b$ are constants.

(a) Given that $gg(2) = 10$ and $f^{-1}(2) = 14$, find the values of $a$ and $b$.

(b) Using these values of $a$ and $b$, find an expression for $gf(x)$ in the form $cx + d$, where $c$ and $d$ are constants.

\[ gf(x) = 3 \left( \frac{1}{2}x - 5 \right) - 2 \]
\[ = \frac{3}{2}x - 17 \]

Solution: (a) $gg(x) = g(3x + b)$

\[ = 3(3x + b) + b \]
\[ = 9x + 4b \]
\[ \therefore gg(2) = 18 + 4b = 10 \text{ new} \]
\[ \Rightarrow b = -2 \checkmark \]

Now $f^{-1}(2) = 14 \Rightarrow f(14) = 2$ (\[ \therefore f(x) = \frac{1}{2}x - a \])
\[ \Rightarrow \frac{1}{2} \times 14 - a = 2 \]
\[ \Rightarrow a = 5 \checkmark \]

(b) Now for $b = -2$ and $a = 5$

$g(x) = 3x - 2$ and $f(x) = \frac{1}{2}x - 5$

\[ \therefore gf(x) = g \left( \frac{1}{2}x - 5 \right) \]
\[ = 3 \left( \frac{1}{2}x - 5 \right) - 2 \]
\[ = \frac{3}{2}x - 17 \checkmark \]
Example 6: The function \( f \) is defined for \( x \in \mathbb{R} \) by, 
\[
f(x) \quad x \mapsto a - 2x
\]
where \( a \) is a constant.

(a) Express \( f \circ f(x) \) and \( f^{-1}(x) \) in terms of \( a \) and \( x \) \(-[4]\)

(b) Given that \( f \circ f(x) = f^{-1}(x) \), find \( x \) in terms of \( a \) \(-[2]\)

Solution (a) 
\[
f(x) = a - 2x \Rightarrow f\circ f(x) = f(a - 2x) = a - 2(a - 2x) = 4x - a \quad ①
\]

New \( f(x) = y = a - 2x \)

Exchange \( x \) and \( y \) \(\Rightarrow x = a - 2y \Rightarrow y = \frac{a - x}{2} \)

\[\therefore f^{-1}(x) = \frac{a - x}{2} \quad ②\]

(b) Given \( f \circ f(x) = f^{-1}(x) \Rightarrow 4x - a = \frac{a - x}{2} \) from ① and ②

\[\Rightarrow 9x = 3a \quad \text{or} \quad x = \frac{a}{3}\]

Example 7: In each of the parts (a), (b) and (c), the graph shown with solid lines has equation \( y = f(x) \). The graph shown with broken lines is a transformation of \( y = f(x) \).

State, in terms of \( f \), the equation of the graph shown with broken line.

Solution:

(a) \( y = f(-x) \) \(\checkmark\)

(b) \( y = 2f(x) \) \(\checkmark\)

(c) \( y = f(x + 4) - 3 \) \(\checkmark\)
Example 8: The function $f$ is defined by $f(x) = -2x^2 + 12x - 3$ for $x \in \mathbb{R}$.

(i) Express $-2x^2 + 12x - 3$ in the form $-2(x + a)^2 + b$, where $a$ and $b$ are constants. \[ -[3] \]

(ii) State the greatest value of $f(x)$.

The function $g$ is defined by $g(x) = 2x + 5$ for $x \in \mathbb{R}$.

(iii) Find the value of $x$ for which $g(f(x)) + 1 = 0$. \[ -[3] \]

Solution (i) $-2x^2 + 12x - 3$

\[
= -2(x^2 - 6x + 3^2 - 9) - 3 \\
= -2(x - 3)^2 + 18 - 3 \\
= -2(x - 3)^2 + 15 \checkmark \\
\]

(ii) Has vertex at $(-3, 15)$.

Greatest value of $f(x) = 15 \checkmark$  
[as $f(x) \leq 15$]

(iii) $g(x) = 2x + 5$ for $x \in \mathbb{R}$

$g(f(x)) = g[-2x^2 + 12x - 3]$  
$= 2(-2x^2 + 12x - 3) + 5$  
$g(x) = -4x^2 + 24x - 1$  
\[ \therefore g(f(x)) + 1 = -4x^2 + 24x - 1 + 1 = 0 \]

or $-4x^2 + 24x = 0$  
$4x[-x + 6] = 0$  
\[ \Rightarrow x = 0 \text{ or } 6 \checkmark \]
Example 9: Functions $f$ and $g$ are defined by,

\[ f: x \to 3x - 2, \quad x \in \mathbb{R} \]

\[ g: x \to \frac{2x+3}{x-1}, \quad x \in \mathbb{R}, \quad x \neq 1 \]

(i) Obtain expressions for $f^{-1}(x)$ and $g^{-1}(x)$, stating the values of $x$ for which $g^{-1}(x)$ is not defined. \[ \text{[4]} \]

(ii) Solve the equation $fg(x) = \frac{7}{3}$. \[ \text{[3]} \]

Solution (i) \[ f(x) = 3x - 2 \quad \text{[0]} \]

or \[ y = 3x - 2 \]

interchange $x$ and $y$

\[ x = 3y - 2 \]

or \[ y = \frac{x+2}{3} \]

hence \[ f^{-1}(x) = \frac{x+2}{3} \quad \text{[3]} \]

Now \[ g(x) = \frac{2x+3}{x-1} \]

or \[ y = \frac{2x+3}{x-1} \]

interchange $x$ and $y$

\[ x = \frac{2y+3}{y-1} \]

\[ \Rightarrow x(y-1) = 2y + 3 \]

\[ xy - x = 2y + 3 \]

\[ x + 2y = x + 3 \]

\[ y(x+2) = x + 3 \]

\[ y = \frac{x+3}{x-2} \]

or \[ g^{-1}(x) = \frac{x+3}{x-2} \quad \text{[3]} \]

or \[ 5x = -40 \]

\[ x = -8 \quad \text{[3]} \]
Example 10: The function \( f \) is defined by \( f(x) = \frac{48}{x-1} \) for \( 3 \leq x \leq 7 \). The function \( g \) is defined by \( g(x) = 2x - 4 \) for \( a \leq x \leq b \), where \( a \) and \( b \) are constant.

(i) Find the greatest value \( y = a \) and the least value of \( b \) which will permit the formation of the composite function \( gf \).

(ii) Find an expression for \( gf(x) \).

(iii) Find an expression for \( (gf)^{-1}(x) \).

Solution:

(i) \( f(x) = \frac{48}{x-1} \) for \( 3 \leq x \leq 7 \)

\[ f(3) = 24, \quad f(7) = 8 \]

Range of \( f(x) \): \( 8 \leq f(x) \leq 24 \)

For \( gf \) to be defined, the domain of \( g(x) \) is same as the range of \( f(x) \), \( g(x) \) is defined for \( 8 \leq x \leq 24 \).

Max \( a = 8 \quad \text{Range of } f(x) \)

and \( \text{Min } b = 24 \quad \text{Domain of } g(x) \)

\[ g(f(x)) = 9 \left( \frac{48}{x-1} \right) = 2 \left( \frac{48}{x-1} \right) - 4 \left( g(f(x)) = 2x - 4 \right) = \left( \frac{100 - 4x}{x-1} \right) \]

(ii) \[ g(f(x)) = 9 \left( \frac{48}{x-1} \right) = 2 \left( \frac{48}{x-1} \right) - 4 \left( g(f(x)) = 2x - 4 \right) = \left( \frac{100 - 4x}{x-1} \right) \]

\[ x = \frac{100 - 4y}{y-1} \]

\[ x(y-1) = 100 - 4y \]

\[ xy - x = 100 - 4y \]

\[ xy + 4y = 100 + x \]

\[ y(x + 4) = 100 + x \]

\[ y = \frac{100 + x}{x + 4} \]

\( (gf)^{-1}(x) = \frac{96 + 1}{x + 4} \)
Example 11: Functions \( f \) and \( g \) are defined by:

\[
f: x \mapsto \frac{3}{2x+1} \quad \text{for } x > 0
\]

\[
g: x \mapsto \frac{1}{x} + 2 \quad \text{for } x > 0
\]

(i) Find the range of \( f \) and the range of \( g \). --[3]

(ii) Find an expression for \( fg(x) \), giving your answer in the form \( \frac{ax}{bx+c} \) where \( a, b \) and \( c \) are integers. --[3]

(iii) Find an expression for \((fg)^{-1}(x)\), giving your answer in the same form as for part (ii) --[3]

Solution:

\( f(x) = \frac{3}{2x+1}, \quad x > 0 \)

Range of \( f \): \( 0 < f(x) < 3 \sqrt{ } \)

And \( g(x) = \frac{1}{x} + 2, \quad x > 0 \)

Range of \( g \): \( g(x) > 2 \)

\( fg(x) = f \left( \frac{1}{x} + 2 \right) = \frac{3}{2\left(\frac{1}{x} + 2\right) + 1} = \frac{3x}{2+5x} \)

\( y = \frac{3x}{2+5x} \)

Interchange \( x \) & \( y \)

\( x = \frac{3y}{2+5y} \)

\( x(5y + 2) = 3y \)

\( 5xy + 2x = 3y \)

\( y(5x-3) = -2x \)

\( y = \frac{2x}{3-5x} \)

\( \therefore (fg)^{-1}(x) = \frac{2x}{3-5x} \)
Example 12: Functions $f$ and $g$ are defined by,
\[ f(x) = 3x^2 + 8x + 1 \quad \text{for } x \in \mathbb{R} \]
\[ g(x) = 2x - k \quad \text{for } x \in \mathbb{R} \]

In case where $k = -1$, find $g'(f(x))$ and solve the equation $g'(f(x)) = 0$.

Solution: for $k = -1$,
\[ g(x) = 2x + 1 \]
\[ g'(x) = 2 \]

Interchange $x$ and $y$,
\[ x = 2y + 1 \]
\[ \Rightarrow y = \frac{x - 1}{2} \]
\[ g^{-1}(x) = \frac{x - 1}{2} \]

Now,
\[ g^{-1}(f(x)) = g^{-1}(2x^2 + 8x + 1) = 0 \]
\[ \Rightarrow \left(\frac{2x^2 + 8x + 1}{2} - 1\right) = 0 \]
\[ \Rightarrow \frac{x^2 + 4x}{2} = 0 \]
\[ \Rightarrow x(x + 4) = 0 \]
\[ \therefore x = 0, -4 \checkmark \]
Example 12. The function $g$ is defined by $g(x) = x^2 - 6x + 7$ for $x > 4$. By first completing the square, find an expression for $g^{-1}(x)$ and state the domain of $g^{-1}$.

$g(x) = x^2 - 6x + 7 = x^2 - 6x + 3^2 - 9 + 7$

$= (x - 3)^2 - 2, \ x > 4$

Let $y = (x - 3)^2 - 2$

$\{ g(x) \geq -1 \}$

Interchange $x$ and $y$

$x = (y - 3)^2 - 2$

or $(y - 3)^2 = x + 2$

$y - 3 = \pm \sqrt{x + 2}$

$y = 3 \pm \sqrt{x + 2}$

$g^{-1}(x) = 3 + \sqrt{x + 2}$

Range of $g^{-1}(x)$

$\{ = \text{domain of } g(x) \}

x > 4$

Range choose + sign

Domain of $g^{-1}(x) = \text{range of } g(x) \geq -1$

Domain of $g^{-1}(x)$ is $x > 4$

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$\rightarrow x \rightarrow x \rightarrow$