Pure Maths 1

Integration
Revision
SP-20 M-20 S-20 M-19 S-19 W-19

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Example 1: A curve has equation \( y = f(x) \). It is given that \( f'(x) = \frac{1}{\sqrt{x+6}} + \frac{6}{x^2} \) and that \( f(3) = 1 \), find \( f(x) \). 

**Solution:** 
\[
\begin{align*}
 f'(x) &= (x+6)^{-\frac{1}{2}} + 6x^{-2} \\
 f(x) &= \int [(x+6)^{-\frac{1}{2}} + 6x^{-2}] \, dx \\
 f(x) &= 2(x+6)^{\frac{1}{2}} - \frac{6}{x} + C
\end{align*}
\]

Given \( f(3) = 1 \) 

\[
1 = 2(3+6)^{\frac{1}{2}} - \frac{6}{3} + C
\]

\[
\Rightarrow C = -3
\]

\[
f(0) = f(x) = 2(x+6)^{\frac{1}{2}} - \frac{6}{x} - 3
\]

Example 2: The diagram shows the curve with equation \( y = x(x-2)^2 \). The minimum point on the curve has coordinate \((a,0)\) and the x-coordinate of its maximum is \(b\), where \(a\) and \(b\) are constants.

(a) state the value of \( a \). 

(b) Calculate the value of \( b \). 

(c) Find the area of the shaded region. 

(d) The gradient, \( \frac{dy}{dx} \), of the curve has a minimum value when \( x = \) \_, calculate the value of \( m \). 

**Solution:** 
\[
y = x(x-2)^2
\]

(a) \( (0,0) \) lies on \( \Rightarrow 0 = a(a-2)^2 \Rightarrow a = 2 \)

(b) \( y = x^3 - 4x^2 + 4x \\
\frac{dy}{dx} = 3x^2 - 8x + 4 \\
(3x-2)(x-2) = 0 \) for stationary point \( \Rightarrow x = 2, \frac{2}{3} \)

\[b = \frac{2}{3} \sqrt{3} \]

(c) \( \text{Area} = \int_0^a y \, dx = \int_{\frac{3}{2}}^2 (x^3 - 4x^2 + 4x) \, dx \\
= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 \right]_0^2 \\
= \frac{4}{3} - \frac{32}{3} + 8 = \frac{4}{3} \]

(d) Gradient \( g = \frac{dy}{dx} = 3x^2 - 8x + 4 \\
\frac{dg}{dx} = 6x-8 \\
\text{for gradient } \frac{dy}{dx} \text{ to be minimum, } \\
\frac{dg}{dx} = 0 \Rightarrow 6x-8 = 0 \Rightarrow x = \frac{4}{3} \)

(e) Min. value of \( \frac{dy}{dx} \) is \( m = \frac{4}{3} \)
Example 3: The diagram shows part of the curve $y = x^2 + 1$. The shaded region enclosed by the curve, the y-axis, and the line $y = 5$ is rotated through 360° about the y-axis. Find the volume obtained. 

Solution: $y = x^2 + 1 \Rightarrow x^2 = y - 1$.

\[
V = \pi \int x^2 \, dy = \pi \int_1^5 (y - 1) \, dy
\]

\[
= \pi \left[ \frac{y^2}{2} - y \right]_1^5
\]

\[
= \pi \left[ \left( \frac{25}{2} - 5 \right) - \left( \frac{1}{2} - 1 \right) \right]
\]

\[
V = \frac{8\pi}{3}
\]

Example 4: The gradient of a curve at the point $(x, y)$ is given by

\[
dy = 2(a+3)^{1/2} - x.
\]

The curve has a stationary point at $(a, 14)$, where $a$ is a positive constant. Find the equation of the curve.

Solution: $\frac{dy}{dx} = 2(a+3)^{1/2} - x \Rightarrow 0$

For stationary point $\frac{dy}{dx} = 0$

\[
2(a+3)^{1/2} - a = 0
\]

\[
= 2(a+3)^{1/2} = a
\]

\[
4(a+3) = a^2
\]

\[
a^2 - 4a - 12 = 0
\]

\[
(a-6)(a+2) = 0
\]

\[
a = 6 \quad \text{or} \quad a = -2
\]

\[
\therefore \quad a = 6 \quad \text{as} \quad a > 0
\]

Stationary point is $(6, 14)$ on the curve.

\[
y = \int (2(a+3)^{1/2} - x) \, dx
\]

\[
y = 2(a+3)^{3/2} - \frac{x^2}{2} + c
\]

\[
y = \frac{4}{3} (a+3)^{3/2} - \frac{x^2}{2} + c
\]

\[
\Rightarrow 14 = \frac{4}{3} (a+3)^{3/2} - \frac{6^2}{2} + c
\]

\[
14 = \frac{4}{3} \times 27 - 18 + c
\]

\[
\Rightarrow c = -4
\]

\[
\text{From (2) equ. of the curve is}
\]

\[
y = \frac{4}{3} (a+3)^{3/2} - \frac{2^2}{2} - 4
\]
Example 5: The diagram shows part of the curve \( y = \frac{8}{x+2} \) and the line \( 2y + x = 8 \), intersecting at points A and B. The point C lies on the curve and the tangent to the curve at C is parallel to AB.

(a) Find, by calculation, the coordinates of A, B and C.

(b) Find the volume generated when the shaded region, bounded by the curve and the line, is rotated through 360° about the x-axis.

Solution: \( y = \frac{8}{x+2} \)  

\[
\begin{align*}
\text{(a)} & \quad \text{line: } 2y + x = 8 \\
& \quad y = \left(\frac{8-x}{2}\right) \quad \text{(2)} \\
& \quad \text{Solving (1) and (2)} \\
& \quad \frac{8}{x+2} = \left(\frac{8-x}{2}\right) \\
& \quad \Rightarrow (x+2)(8-x) = 8 \\
& \quad x^2 - 6x = 0 \\
& \quad x(x-6) = 0 \\
& \quad x = 0 \quad \text{or} \quad x = 6
\end{align*}
\]

\[
\begin{align*}
& \quad \text{Region 1: } y = 4 \quad \text{and} \quad x = 6 \\
& \quad \therefore A(0,4) \quad \text{and} \quad B(6,1) \\
& \quad \text{Gradient of } AB = 1 - 4 = -\frac{3}{6} = -\frac{1}{2} \quad \text{(5)} \\
& \quad \text{Gradient of the tangent at } C, \quad \frac{dy}{dx} = -\frac{1}{2} \quad \text{(6)} \\
& \quad \text{Differentiating: } \frac{dy}{dx} = -\frac{8}{(x+2)^2} = \frac{-1}{2} \quad \text{from (5)} \\
& \quad \Rightarrow (x+2)^2 = 16 \\
& \quad x + 2 = \pm 4 \\
& \quad x = 2 \quad \text{or} \quad x = -6 \\
& \quad \text{Region 2: } y = 2 \\
& \quad C(3,2)
\end{align*}
\]

\[
\begin{align*}
& \quad \text{Volume under the line} \\
& \quad = \pi \int_0^6 \left[4 - \frac{1}{2}x\right]^2 \, dx \\
& \quad = \pi \int_0^6 \left[16 + \frac{1}{2}x^2 - 4x\right] \, dx \\
& \quad = \pi \left[16x + \frac{x^3}{3} - 2x^2\right]_0^6 \\
& \quad = \pi \left[96 - 0\right] = 96\pi \\
& \quad \text{Area under the curve} \\
& \quad = \pi \int_0^6 y^2 \, dx = \pi \int_0^6 \left(\frac{x}{x+2}\right)^2 \, dx \\
& \quad = \pi \left[-\frac{6x}{x+2}\right]_0^6 \\
& \quad = \pi \left[-36 - (-32)\right] \\
& \quad = 2\pi \quad \text{(5)} \\
& \quad \text{Requisite shaded area} \\
& \quad = 96\pi - 2\pi \left[\text{from (5)}\right] \\
& \quad = 94\pi \quad \text{(6)}
\end{align*}
\]
Example 6: The diagram shows part of the curve \( y = \frac{6}{x} \). The points \((1,6)\) and \((3,2)\) lies on the curve. The shaded region is bounded by the curve and the lines \( y = 6 \) and \( x = 1 \).

(a) Find the volume generated when the shaded region is rotated through 360° about the y-axis.

(b) The tangent to the curve at a point \( x \) is parallel to line \( y + 2x = 0 \), show that \( x \) lies on the line \( y = 2x \).

Solution: Volume by rotating the curve about the y-axis.

\[
\text{Volume of Cylinder by rotating the line } x = 1 \quad \pi \int_0^2 (6)^2 \, dy
\]

\[
\pi \int_0^2 36 \, dy = \pi \left[ \frac{36}{2} \right]^2
\]

\[
= \pi \left[ \frac{-36}{2} \right]^2 = \pi \left[ -6 - (-18) \right] = 12\pi
\]

Volume of Cylinder by rotating the line \( x = 1 \)

\[
\pi \int_0^2 (6)^2 \, dy = \pi \int_0^6 \frac{6}{x} \, dx
\]

\[
\pi \int_0^2 \frac{6}{x} \, dx = 4\pi
\]

Required Volume by rotating the shaded region about y-axis

\[
\int_0^2 = 12\pi - 4\pi = 8\pi
\]

(b) \( x = \frac{6}{y} \) or \( y = \frac{6}{x} \)

Gradient of the line \( y = -2 \) --

Gradient of tangent to the curve \( \frac{dy}{dx} = \frac{-6}{x^2} \)

Curve is \( y = \frac{6}{x} \)

\[
\frac{dy}{dx} = \frac{-6}{x^2}
\]

\[
\frac{-6}{x^2} = -2
\]

\[
x = \sqrt{3}
\]

\[
\{ x = \sqrt{3}, y = \sqrt{3} \}
\]

\[
(\sqrt{3}, 2\sqrt{3}) \text{ lies on } y = 2x
\]
Example 7: The equation of a curve is such that \( \frac{dy}{dx} = 3x^{3/2} - 3x^{1/2} \),

It is given that the point \((4, 7)\) lies on the curve. Find the equation of the curve.

Solution:

\[
\frac{dy}{dx} = 3x^{3/2} - 3x^{1/2} \\
\Rightarrow y = \int \left(3x^{3/2} - 3x^{1/2}\right) \, dx \\
= 3x^{3/2} - 3x^{1/2} + c
\]

or \( y = 2x^{3/2} - 6x^{1/2} + c \quad (1) \)

Given \((4, 7)\) lies on the curve \( \Rightarrow 7 = 2(4)^{3/2} - 6(4)^{1/2} + c \)

\( \Rightarrow 7 = 16 - 12 + c \Rightarrow c = 3 \)

\( \therefore \) from \( (1) \), Eq. equ of curve: \( y = 2x^{3/2} - 6x^{1/2} + 3 \)

Example 8: The diagram shows part of the curve with equation \( y = x^3 - 2bx^2 + b^2x \) and the line \( OA \), where \( A \) is the maximum point on the curve. The \( x \)-coordinate of \( A \) is \( a \) and the curve has a minimum point at \((b, 0)\), where \( a \) and \( b \) are positive constants.

(a) Show that \( b = 3a \)

(b) Show that the area of the shaded region between the line and the curve is \( ka^4 \), where \( k \) is a fraction to be found.

Solution:

(a) has been done in differentiation revision; \( b = 3a \).

(b) Shaded region area = area under the curve - area under the line - \( \square \)

Area under the curve:

\[
\int_{0}^{a} \left(x^3 - 2bx^2 + b^2x\right) \, dx
\]

\[
A = \left[ \frac{x^4}{4} - \frac{2bx^3}{3} + \frac{b^2x^2}{2}\right]_{0}^{a}
\]

\[
= \left(\frac{a^4}{4} - \frac{2baa^2}{3} + \frac{b^2a^2}{2}\right) - 0
\]

\[
= \frac{1}{4} a^4 - \frac{2}{3} ba^2 + \frac{1}{2} b^2 a^2
\]

Now at \( x = a \), \( y = a^3 - 6a^2 + 9a^3 = 4a^3 \)

Area under the curve at \( (a, 4a^3) \):

\[
A(a, 4a^3) = \frac{1}{4} a^4 - \frac{2}{3} ba^2 + \frac{1}{2} b^2 a^2
\]

Area under the line \( OA \) (AP):

\[
\frac{1}{2} x \times AP
\]

\[
= \frac{1}{2} a \times 4a^3
\]

\[
= 2a^4
\]

Area of the shaded region:

\[
= \frac{1}{4} a^4 - \frac{2}{3} ba^2 + \frac{1}{2} b^2 a^2 - 2a^4
\]

\[
= \frac{1}{4} a^4 - \frac{2}{3} ba^2 + \frac{1}{2} b^2 a^2 - 2a^4
\]

\[
= \frac{3}{4} a^4
\]
Example 9: A curve has equation \( y = f(x) \), passes through the points (0, 2) and (3, -1). It is given that \( f'(x) = kx^2 - 2x \), where \( k \) is a constant. Find the value of \( k \). ---[5]

Solution: 
\[ f'(x) = kx^2 - 2x \Rightarrow f(x) = \int (kx^2 - 2x) \, dx \]
\[ \Rightarrow y = k \frac{x^3}{3} - x^2 + c \quad - \mathbb{D} \]

(0, 2) lies on \( D \Rightarrow 2 = c \); and (3, -1) also lies on \( D \)

\[ \Rightarrow -1 = 9k - 9 + a \quad [; c = 2] \]

\[ \Rightarrow k = \frac{a}{9} \Rightarrow k = \frac{3}{2} \checkmark \]

Example 10: The diagram shows part of the curve with equation \( y = \sqrt{x^3 + x^2} \). The shaded region is bounded by the curve, the x-axis, and the line \( x = 3 \).
Find showing all necessary working, the volume obtained when the shaded region is rotated through \( 360^\circ \) about the x-axis.

\[ \text{Solution: } V = \pi \int y^2 \, dx = \pi \int_0^3 (x^3 + x^2) \, dx \]

\[ = \pi \left[ \frac{x^4}{4} + \frac{x^3}{3} \right]_0^3 \]

\[ = \pi \left[ \left( \frac{81}{4} + \frac{27}{3} \right) - 0 \right] \]

\[ = \frac{117\pi}{4} \checkmark \]
Example 11: A curve for which \( \frac{d^2y}{dx^2} = 2x - 5 \), has a stationary point at \((3, 6)\). Find the equation of the curve.

**Solution:**

\[
\frac{d^2y}{dx^2} = 2x - 5 \implies \frac{dy}{dx} = \int (2x - 5) \, dx
\]

\[
= x^2 - 5x + C = 0 \text{ for stationary point.}
\]

\[
\therefore \left(\frac{dy}{dx}\right)_{(3, 6)} = 9 - 15 + C = 0 \implies C = 6
\]

\[
\therefore \frac{dy}{dx} = x^2 - 5x + 6
\]

Integrate:

\[
y = \int (x^2 - 5x + 6) \, dx = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + d
\]

Curve passes through \((3, 6)\):

\[
6 = \frac{27}{3} - \frac{45}{2} + 18 + d \implies d = \frac{3}{2}
\]

\[
\therefore \text{equation of curve: } y = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + \frac{3}{2}
\]

Example 12: The diagram shows part of the curve \( y = \frac{3}{\sqrt{1 + 4x}} \) and a point \((2, 1)\) lying on the curve. The normal to the curve at \(P\) intersects the \(x\)-axis at \(Q\).

(i) Show that the \(x\)-coordinate of \(Q\) is \(4/9\).

(ii) Find showing all necessary working, the area of the shaded region.

**Solution:**

\[
y = \frac{3}{\sqrt{1 + 4x}} \quad \text{①}
\]

\[
\frac{dy}{dx} = \frac{3}{2} \cdot \frac{1}{(1+4x)^{3/2}} \cdot 4 = 6(1 + 4x)^{-3/2}
\]

\[
\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = \frac{-3}{2} \left(9\right)^{-3/2} = \frac{-2}{9}
\]

\[
\therefore \text{gradient of the normal at } P = \frac{2}{3}
\]

**Equation of Normal at \(P(2, 1)\):**

\[
y - 1 = \frac{2}{3} (x - 2)
\]

For \(y = 0\):

\[
x = \frac{4}{9} \implies Q\left(\frac{4}{9}, 0\right)
\]

**Draw** \(PR \perp x\)-axis.

**Area of the shaded region:**

\[
= \text{area under the curve} - \text{area of } \triangle PQR \quad \text{②}
\]

Area under the curve:

\[
\int_{0}^{4/9} \frac{3}{\sqrt{1 + 4x}} \, dx = \left[ \frac{3\sqrt{1 + 4x}}{2} \right]_{0}^{4/9}
\]

Area of \(\triangle PQR = \frac{1}{2} \times QR \times PR
\]

Area of the shaded region:

\[
= 3 - \frac{4}{9} = \frac{26}{9}
\]

\[
\therefore \text{Area} = \frac{26}{9}
\]

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Example 13: A curve is such that \( \frac{dy}{dx} = \frac{x^3 - 4}{x^2} \). The point \( P(2, 9) \) lies on the curve. Find the equation of the curve.

Solution: \( \frac{dy}{dx} = \frac{x^3 - 4}{x^2} \)

Integrate \( y = \int \left( \frac{x^3 - 4}{x^2} \right) \, dx \)

\[ y = \frac{x^4}{4} + \frac{4}{x} + C \quad \cdots \, 0 \]

At point \( P(2, 9) \) lies on the curve \( \Rightarrow 9 = \frac{2^4}{4} + \frac{4}{2} + C \Rightarrow C = 3 \)

Equation of the curve: \( y = \frac{x^4}{4} + \frac{4}{x} + 3 \) 

Example 14: The diagram shows part of the curve \( y = \sqrt{4x+1} + \frac{9}{\sqrt{4x+1}} \) and the minimum point \( M \).

(i) Find expressions for \( \frac{dy}{dx} \) and \( \frac{dy}{dx} \)

(ii) Find the coordinates of \( M \).

The shaded region is bounded by the curve, the \( y \)-axis and the line through \( M \) parallel to \( x \)-axis.

(iii) Find, showing all necessary working, the area of the shaded region.

Solution: \( y = \sqrt{4x+1} + \frac{9}{\sqrt{4x+1}} \)

\[ \frac{dy}{dx} = \frac{1}{2} (4x+1)^{-\frac{1}{2}} + \frac{9}{2} (4x+1)^{-\frac{3}{2}} \]

\[ = \frac{2x - 16}{(4x+1)^{\frac{3}{2}}} \quad \cdots \, 2 \]

\[ \Rightarrow \text{Area } = \int_0^2 y \, dx - (2 \times 6) \]

\[ = \left[ \frac{(4x+1)^{\frac{3}{2}} + \frac{9}{2} (4x+1)^{\frac{1}{2}}} {\frac{3}{2}} \right]_0^2 - 12 \]

\[ = \left[ \left( \frac{3}{2} + \frac{3}{2} \right) - \left( \frac{1}{2} + \frac{3}{2} \right) \right] - 12 \]

\[ = \frac{4}{3} - 12 \]

\[ = \frac{4}{3} - 12 \]

\[ = \frac{1}{3} \]

\[ A = \frac{1}{3} \left( \text{or} \ 1.33 \right) \]
Example 15: A curve is such that \( \frac{dy}{dx} = 3x^2 + 2x + b \). The curve has stationary points at \((-1, 2)\) and \((3, k)\). Find the value of the constants \( a, b \) and \( k \).

Solution: \( \frac{dy}{dx} = 3x^2 + 2x + b \) — (1)

Stationary point at \((-1, 2)\) \( \Rightarrow f'(-1) = 3 - a + b = 0 \Rightarrow a + b = 3 \) — (2)

Also at \((3, k)\) \( \Rightarrow f'(3) = 27 + 3a + b = 27 + 3a + b = 27 - a \) — (3)

Solve \( (2) \) \& \( (3) \) \( \Rightarrow a = -6 \) and \( b = -9 \)

Now integrate \( y = \int (3x^2 - 6x - 9) \, dx \): \( \therefore a = -6 \)

Equation for the curve \( y = x^3 - 3x^2 - 9x + C \) — (4)

Now point \((-1, 2)\) lies on the curve \( \Rightarrow 2 = -1 - 3 + 9 + C \Rightarrow C = -3 \)

But \( C = -3 \) from (3) \( y = x^3 - 3x^2 - 9x - 3 \) — (5)

Now point \((3, k)\) lies on it \( \Rightarrow k = 27 - 27 - 27 - 3 \Rightarrow k = -30 \)

Example 16: The diagram shows part of the curve \( y = (3x + 4)^2 \) and the tangent to the curve at the point \( A \). The \( x \)-coordinate of \( A \) is 4.

(i) Find the equation of the tangent to the curve at \( A \).

(ii) Find the area of the shaded region.

Solution: \( y = (3x + 4)^2 \) — (1)

(i) \( \frac{dy}{dx} = \frac{1}{2} (3x + 4)^1 \cdot 3 = \frac{3}{2} (3x + 4) \) — (2)

\( \frac{dy}{dx} = \frac{3}{2} \) \( \Rightarrow \) gradient

\( \therefore \) Equation of tangent at \((4, y)\)

\( y - y = \frac{3}{2} (x - 4) \)

\( \therefore \) Gradient

\( \therefore \) Tangent intersects \( y \)-axis at \( \frac{2}{3} \)

\( \therefore \) From \( (3) \) \( x = 0 \Rightarrow y = \frac{5}{2} \) V

(ii) Area of the shaded region.

= Area Under the Tangent

= Area Under the Curve

Area Under the Tangent

\[ \int (3x + 4)^2 \, dx = \left[ \frac{1}{3} (3x + 4)^3 \right]_4 \]

\[ = \frac{288 - 16}{9} = 12 \]

= Shaded Area = \( 13 - \frac{125}{9} \)

= \( \frac{5}{9} \) (or \( 0.556 \))
Example 17: An increasing function, $f$, is defined for $x > n$, where $n$ is an integer. It is given that $f'(x) = x^2 - 6x + 8$. Find the least possible value of $n$.

Solution: for increasing $f'(x) > 0 \Rightarrow x^2 - 6x + 8 > 0 \quad (x-2)(x-4) \geq 0$

\[
\begin{array}{c|c}
2 & 4 \\
\hline
2 & 4 \\
\end{array}
\]

\[\therefore \text{least value of } n = 4\sqrt{\]

Example 18: A curve for which $\frac{dy}{dx} = (5x-1)^{\frac{1}{2}} - 2$ passes through the point $(2, 3)$.

(i) Find the equation of the curve.

(ii) Find $\frac{d^2y}{dx^2}$.

(iii) Find the coordinates of the stationary point on the curve and, showing all necessary working, determine the nature of this stationary point.

Solution:

\[
\frac{dy}{dx} = (5x-1)^{\frac{1}{2}} - 2 = 0
\]

\[
\text{from } 0 \rightarrow (5x-1)^{\frac{1}{2}} - 2 = 0
\]

\[
5x-1 = 4
\]

\[
x = 1
\]

\[
\begin{align*}
\text{from } (2, 3) & \Rightarrow \\
3 & = \frac{2}{5} \cdot 2^2 - 4 + c \Rightarrow c = \frac{17}{5}
\end{align*}
\]

\[
\begin{align*}
\text{from III} & \Rightarrow y = \frac{16}{15} - \frac{9}{5} + \frac{17}{5} = \frac{37}{15}
\end{align*}
\]

\[
\begin{align*}
\text{Stationary point } (1, \frac{37}{15})
\end{align*}
\]

\[
\begin{align*}
\text{(i) Integration: } y & = \int (5x-1)^{\frac{1}{2}} - 2 dx \\
& = (5x-1)^{\frac{3}{2}} - 2x + c
\end{align*}
\]

\[
\begin{align*}
\text{from } 0 & \rightarrow (5x-1)^{\frac{3}{2}} - 2x + c = \frac{17}{5}
\end{align*}
\]

\[
\begin{align*}
\text{from III} & \Rightarrow y = \frac{16}{15} - \frac{9}{5} + \frac{17}{5} = \frac{37}{15}
\end{align*}
\]

\[
\begin{align*}
\text{Stationary point } (1, \frac{37}{15})
\end{align*}
\]

\[
\begin{align*}
\text{(ii) Supp } 0 & \frac{d^2y}{dx^2} = \frac{1}{2} (5x-1)^{-\frac{1}{2}} x \\
& = \frac{5}{2} (5x-1)^{-\frac{1}{2}} \quad (d^2y)_{x=1} = \frac{5}{2} \cdot \frac{1}{2} = \frac{5}{4} > 0
\end{align*}
\]

\[
\therefore \text{Minimum at } (1, \frac{37}{15})
\]
Example 19: The diagram shows a shaded region bounded by the y-axis, the line \( y = -1 \) and the part of the curve \( y = x^2 + 4x + 3 \) for which \( x > -2 \).

(i) Express \( y = x^2 + 4x + 3 \) in the form \( y = (x+a)^2 + b \), where \( a \) and \( b \) are constants. Hence, for \( x > -2 \), express \( x \) in terms of \( y \). \(-[4]\)

(ii) Hence, showing all necessary working, find the volume obtained when the shaded region is rotated through 360° about the y-axis. \([6]\)

Solution: \( y = x^2 + 4x + 3 \)

(i) \[
y = (x+2)^2 - 1 \tag{1}
\]

Now \( (x+2)^2 = y+1 \)

\[
x + 2 = \pm \sqrt{y+1} \sqrt{y+1} \text{ for } x \geq -2
\]

(ii) \[
x^2 = \left[-2 + (y+1)^{\frac{1}{2}}\right]^2
\]

\[
= 4 + (y+1) - 4 \left(y+1\right)^{\frac{1}{2}}
\]

\[
V = \pi \int_{-1}^{3} x^2 \, dy
\]

\[
= \pi \int_{-1}^{3} (5y+y^2-4(y+1)^{\frac{3}{2}}) \, dy
\]

\[
= \pi \left[ 5y + \frac{y^2}{3} - 4(y+1)^{\frac{3}{2}} \right]_{-1}^{3}
\]

\[
= \pi \left[ 5\cdot3 + \frac{9}{3} - \left(-5 + \frac{1}{3}\right) \right] - \pi \left[ 5\cdot(-1) + \frac{1}{3} - \left(-5 + \frac{1}{3}\right) \right]
\]

\[
= 8\pi \left( \text{or } 8.38 \right) \checkmark
Example 20: A curve is such that \( \frac{dy}{dx} = \frac{k}{\sqrt{x}} \) where \( k \) is a constant. The points \( P(1, -1) \) and \( Q(4, 4) \) lie on the curve. Find the equation of the curve.

Solution: \( \frac{dy}{dx} = k \sqrt{x} \) \( \quad (1) \)

Integrate \( y = \int k \sqrt{x} \, dx \)
\[ = \frac{kx^{\frac{3}{2}}}{\frac{3}{2}} + C \]

or \( y = 2k \sqrt{x} + C \) \( \quad (2) \)

(1) \( P(1, -1) \) lies on \( y \) \[ -1 = 2k + C \] \( \quad (3) \)

(1) \( Q(4, 4) \) lies on \( y \) \[ 4 = 4k + C \] \( \quad (4) \)

Solving (3) \( \Rightarrow k = \frac{1}{2}, C = \frac{1}{2} \)

Hence from (2) \( \text{equation of curve } y = 2 \times \frac{1}{2} \sqrt{x} - \frac{1}{2} \Rightarrow y = 5 \sqrt{x} - 6 \)

Example 21: The diagram shows part of the curve \( y = 1 - \frac{4}{(2x+1)^2} \).

The curve intersects the \( x \)-axis at \( A \). The normal to the curve at \( A \) intersects the \( y \)-axis at \( B \).

(i) Obtain expressions for \( \frac{dy}{dx} \) and \( \int y \, dx \) \( \quad ([4]) \)

(ii) Find the coordinates of \( B \) \( \quad ([4]) \)

(iii) Find the area of the shaded region \( \quad ([4]) \)

Solution: \( y = 1 - \frac{4}{(2x+1)^2} \) \( \quad (1) \)

(i) \( \frac{dy}{dx} = \frac{2 \times (-4)}{(2x+1)^3} \times (2x+1) \)
\[ = \frac{-4}{(2x+1)^3} \]

\[ \int y \, dx = \int \left(1 - \frac{4}{(2x+1)^2}\right) \, dx \]
\[ = x + 2(2x+1)^{-1} + C \] \( \quad (3) \)

(ii) At \( A, y = 0 \) \[ \Rightarrow x = \frac{1}{2}, A \left(\frac{1}{2}, y\right) \]

from (3) \( \frac{dy}{dx} x = \frac{1}{2} \Rightarrow \text{Gradient of Normal} = -\frac{1}{2} \)

Equation of Normal at \( A \left(\frac{1}{2}, y\right) \)
\[ y - 0 = -\frac{1}{2} \left(x - \frac{1}{2}\right) \]

Intersect \( y \)-axis, put \( x = 0 \)
\[ y = \frac{1}{4} \]

\( B \left(0, \frac{1}{4}\right) \)

Area of shaded area = area of \( \triangle AOB \)

Area between curve and \( x \)-axis:
\[ A_1 = \int_0^{1/2} \left(1 - \frac{4}{(2x+1)^2}\right) \, dx \]
\[ = \left[2x - \frac{2}{2x+1}\right]_0^{1/2} = \frac{1}{2} \]

\( [A_1] = \frac{1}{2} \)

Area of \( \triangle AOB = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{16} \)

Area of shaded region
\[ A \triangle AOB = \frac{1}{2} + \frac{1}{16} = \frac{9}{16} \]
Example 22: A function \( f \) is defined for \( x > \frac{1}{2} \) and is such that,
\[
f'(x) = 3(2x-1)^{\frac{1}{2}} - 6.
\]
It is now given that \( f(1) = -3 \). Find \( f(x) \).

Solution: 
\[
f'(x) = 3(2x-1)^{\frac{1}{2}} - 6
\]
Integrating, 
\[
f(x) = \int (3(2x-1)^{\frac{1}{2}} - 6) \, dx
\]
\[
= 3(2x-1)^{\frac{3}{2}} - 6x + C
\]
\[
\Rightarrow f(x) = (2x-1)^{\frac{3}{2}} - 6x + 2 - C
\]
\[
-3 = 1 - 6 + C \Rightarrow C = 2
\]
\[
\therefore f(x) = (2x-1)^{\frac{3}{2}} - 6x + 2
\]

Example 23: The diagram shows part of the curve \( y = (x-1)^{\frac{3}{2}} + 2 \), and the lines \( x = 1 \) and \( x = 3 \). The point \( A \) on the curve has coordinates \( (2,3) \). The normal to the curve at \( A \) crosses the line \( x = 1 \) at \( B \).

(i) Show that the normal \( AB \) has equation \( y = \frac{1}{2} x + 2 \).

(ii) Find the volume of revolution obtained when the shaded region is rotated through \( 360^\circ \) about the \( x \)-axis.

Solution: 
\[
y = (x-1)^{\frac{3}{2}} + 2
\]
(i) 
\[
dy\![x] = 2(x-1)^{-\frac{1}{2}}
\]
\[
\left(\frac{dy}{dx}\right)\!_{x=2} = -2,
\]
\[
\therefore \text{gradient of normal at } A \left(\frac{1}{2}\right).
\]
Equation of normal at \( A(2,3) \):
\[
y - 3 = \frac{1}{2}(x - 2) \Rightarrow y = \frac{1}{2}x + 2
\]
(ii) 
\[
V = \text{Volume by revolving line} \left( \begin{array}{c}
\text{at } x = 1
\text{at } x = 3
\end{array} \right)
\]
\[
V_1 = \pi \int_{0}^{1} y^2 \, dx = \pi \int_{0}^{1} \left( (x-1)^{\frac{3}{2}} + 2 \right)^2 \, dx
\]
\[
= \pi \int_{0}^{1} \left( (x-1)^{\frac{3}{2}} + 4(x-1)^{\frac{1}{2}} + 4 \right) \, dx
\]
\[
= \pi \left[ \frac{2}{3} (x-1)^{\frac{5}{2}} - 4(x-1)^{\frac{3}{2}} + 4x \right]_{0}^{1}
\]
\[
= \pi \left[ - \frac{1}{3} + 8 - 0 \right]
\]
\[
= \pi \left[ -\frac{1}{3} + 8 \right]
\]
\[
V_2 = \pi \left[ \frac{15}{2} \right] = \frac{15}{2} \pi
\]

Required Volume: 
\[
V = V_1 + V_2 = \frac{91}{12} \pi + \frac{15}{4} \pi = \frac{332}{48} \pi
\]

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