Pure Math 1

Quadratics
Revision

Suresh Goel
(former director)
Alliance World School,
Noida, Delhi NCR,
INDIA.
Example 1: The equation of a line is \( y = mx + c \), where \( m \) and \( c \) are constants, and the equation of a curve is \( xy = 16 \).

(a) Given that the line is a tangent to the curve, express \( m \) in terms of \( c \).

(b) Given instead that \( m = -4 \), find the set of values of \( c \) for which the line intersects the curve at two distinct points.

Solution (a)

\[ y = mx + c \quad \text{(1)} \]

Curve: \( xy = 16 \quad \text{(2)} \)

From (1) and (2) \( x(mx + c) = 16 \)

\[ \Rightarrow mx^2 + cx - 16 = 0 \quad \text{(3)} \]

For line to be tangent to curve:

\[ b^2 - 4ac = 0 \]

\[ c^2 - 4 \times m \times (-16) = 0 \]

\[ \Rightarrow c^2 + 64m = 0 \]

\[ \Rightarrow m = -\frac{c^2}{64} \quad \text{\( \checkmark \)} \]

(b) Now \( y = -4x + c \quad \text{(4)} \) for \( m = -4 \)

\[ xy = 16 \quad \text{(2)} \]

\[ \Rightarrow x(-4x + c) = 16 \]

\[ \Rightarrow -4x^2 + cx - 16 = 0 \]

\[ \Rightarrow 4x^2 - cx + 16 = 0 \quad \text{(5)} \]

The line intersects the curve (2) in two different points \( \Rightarrow b^2 - 4ac > 0 \)

From (5) \( \Rightarrow c^2 - 4 \times 4 \times 16 > 0 \)

\[ \frac{c^2 - (16)^2}{4} > 0 \]

\[ \frac{c^2}{16} > (16)^2 \]

\[ \Rightarrow c > 16 \text{ or } c < -16 \]

(\( : x > a^2, a > 0 \) \( \rightarrow \) \( x < -a \) \( \rightarrow \) \( x > a \) )
Example: The equation of a curve is \( y = 2x^2 + kx + k - 1 \), where \( k \) is a constant.

(a) Given that the line \( y = 2x + 3 \) is a tangent to the curve, find the value of \( k \). 

It is now given that \( k = 2 \). 

(b) Express the equation of the curve in the form \( y = a(x+a)^2 + b \), where \( a \) and \( b \) are constants, and state the coordinates of the vertex of the curve.

--- [3]

Solution: Curve: \( y = 2x^2 + kx + k - 1 \)

(a) Line: \( y = 2x + 3 \)

From (a) \( \) 
\[ 2x^2 + kx + k - 1 = 2x + 3 \]
\[ \Rightarrow 2x^2 + (k-2)x + (k-4) = 0 \]

From line (a) be tangent to curve (b). 

Line should intersect exactly at one point \( k \).

For \( b^2 - 4ac = 0 \)

From (b) \( (k-2)^2 - 4 \cdot 2 \cdot (k-4) = 0 \)
\[ \Rightarrow k^2 - 12k + 36 = 0 \]
\[ \Rightarrow (k-6)^2 = 0 \]
\[ \Rightarrow k = 6 \]

(b) Now \( k = 2 \)

From (a) Curve: \( y = 2x^2 + 2x + 1 \)

or \( y = 2 \left[ x^2 + x + \frac{1}{2} \right] \)
\[ = 2 \left[ x^2 + x + \left( \frac{1}{2} \right)^2 - \frac{1}{4} + \frac{1}{2} \right] \]
\[ = 2 \left[ (x+\frac{1}{2})^2 + \frac{1}{2} \right] \]
\[ = 2(x+\frac{1}{2})^2 + \frac{1}{2} \]
\[ \therefore \text{Vertex } (-\frac{1}{2}, \frac{1}{2}) \]
Example 3: Find the set of values of m for which the line with equation \( y = mx + 1 \) and the curve with equation \( y = 3x^2 + 2x + 4 \) intersect at two distinct points.

Solution: Line: \( y = mx + 1 \) \( \quad \) (1)
Curve: \( y = 3x^2 + 2x + 4 \) \( \quad \) (2)

To find the point of intersection from (1) and (2)

\[ 3x^2 + 2x + 4 = mx + 1 \]

or \[ 3x^2 + (2-m)x + 3 = 0 \] \( \quad \) (3)

from (1) and (2) intersect in two distinct points,

\[ b^2 - 4ac > 0 \]

\[ (2-m)^2 - 4 \cdot 3 \cdot 3 > 0 \]

\[ (2-m)^2 - 36 > 0 \] \( \quad \) (4)

\[ (2-m)^2 > 36 \]

\[ 2-m > 6 \text{ or } 2-m < -6 \]

\[ \Rightarrow m < -4 \text{ or } m > 8 \]

Alternate method to solve (4)

\[ (x-a)(x-b) > 0 \]

\[ (2-m)^2 - 36 > 0 \]

\[ m^2 - 4m + 4 - 36 > 0 \]

\[ m^2 - 4m - 32 > 0 \]

\[ (m-8)(m+4) > 0 \]

roots of eqn: 8, -4

\[ m > 8 \quad \text{or} \quad m < -4 \]

Note: \( a > b \)
Example 4: The line $4y = x + c$, where $c$ is a constant, is a tangent to the curve $y^2 = x + 3$ at the point $P$ on the curve.

(i) Find the value of $c$.  
(ii) Find the coordinates of $P$. 

Solution:

(i) Line: $4y = x + c$  
Curve: $y^2 = x + 3$  
From (i) and (ii) $y^2 = 4y - c + 3$  
Or $y^2 - 4y + c - 3 = 0$  
For line to be tangent to the curve $b^2 = 4ac = 0$ in (ii)  
$$(-4)^2 - 4 \times 1 \times (c-3) = 0$$  
$$16 - 4c + 12 = 0 \Rightarrow 4c = 28 \Rightarrow c = 7 \checkmark$$

(ii) Let $c = 7$ in (ii) to find $P$.  
$$y^2 - 4y + (7 - 3) = 0$$  
$$y^2 - 4y + 4 = 0$$  
$$(y - 2)^2 = 0$$  
$$y = 2 \checkmark$$  
From (i) $x = 4y - 7 = 1$  
$$x = 1$$  
$\therefore P(1, 2) \checkmark$
Example 5: The function \( f(x) = x^2 - 4x + 8 \) for \( x \in \mathbb{R} \).

(i) Express \( x^2 - 4x + 8 \) in the form \((x - a)^2 + b\). 

\[ (i) \quad x^2 - 4x + 8 = (x^2 - 4x + 4) + 4 = (x - 2)^2 + 4 \] 

(ii) Hence find the set of values of \( x \) for which \( f(x) < 9 \), giving your answer in exact form.

Solution: \( f(x) = x^2 - 4x + 8 \)

\[ (i) \quad x^2 - 4x + 8 = (x^2 - 4x + 4) + 4 = (x - 2)^2 + 4 \]

\[ (ii) \quad f(x) < 9 \]

\[ \Rightarrow \quad x^2 - 4x + 8 < 9 \]

\[ \Rightarrow \quad (x - 2)^2 + 4 < 9 \]

\[ \Rightarrow \quad (x - 2)^2 < 5 \]

\[ \Rightarrow \quad (x - 2)^2 < (\sqrt{5})^2 \]

\[ \Rightarrow \quad \sqrt{5} < x - 2 < \sqrt{5} \]

\[ \Rightarrow \quad 2 - \sqrt{5} < x < 2 + \sqrt{5} \]
Example 6: A straight line has gradient \( m \) and passes through the point \((0, -2)\). Find the two values of \( m \) for which the line is tangent to the curve \( y = x^2 - 2x + 7 \) and, for each value of \( m \), find the coordinates of the point where the line touches the curve.

Solution: Equation of line passing through \((0, -2)\) and gradient \( m \),

\[
\text{line: } y = mx - 2 \quad \text{--- 1}
\]

\[
\text{curve: } y = x^2 - 2x + 7 \quad \text{--- 2}
\]

\(\text{From 1 and 2: } x^2 - 2x + 7 = mx - 2\)

\(\Rightarrow x^2 - (2 + m)x + 9 = 0 \quad \text{--- 3}\)

For line 1 to be tangent to the curve 2,"} b^2 - 4ac = 0

\[
(2 + m)^2 - 4 \cdot 1 \cdot 9 = 0
\]

\[
(2 + m)^2 = 36
\]

\(\Rightarrow 2 + m = \pm 6 \Rightarrow m = 4 \text{ or } -8\)

Now for \( m = 4 \), from 3,

\[
x^2 - 6x + 9 = 0
\]

\[
(x - 3)^2 = 0 \Rightarrow x = 3, \quad y = 4 \cdot 3 - 2 \quad \text{Point \( (3,10)\)}
\]

Again for \( m = -8 \)

\[
x^2 + 6x + 9 = 0 \Rightarrow (x + 3)^2 = 0 \Rightarrow x = -3
\]

\(\text{From 1 for } m = -8, \quad y = -8 \cdot -3 - 2
\]

\(x = -3 \Rightarrow y = 22 \quad \text{Point \((-3, 22)\)}

\therefore \text{ required points } \((-3, 22) \quad \text{and } \(3, 10)\)
Example 7: Functions $f$ and $g$ are defined by:

\[ f(x) = 2x^2 + 8x + 1 \quad \text{for} \quad x \in \mathbb{R} \]
\[ g(x) = 2x - k \quad \text{for} \quad x \in \mathbb{R} \]

where $k$ is a constant.

(i) Find the value $k$ for which the line $y = g(x)$ is a tangent to the curve $y = f(x)$.

(ii) In case where $k = -9$, find the set of values of $x$ for which $f(x) < g(x)$.

Solution: line: $y = 2x - k \quad \text{(1)}$

curve $f(x) = 2x^2 + 8x + 1 \quad \text{(2)}$

from (1) \& (2)

\[ 2x^2 + 8x + 1 = 2x - k \]

\[ 
\Rightarrow 2x^2 + 6x + (1 + k) = 0 \quad \text{(3)}
\]

For line (1) tangent to curve (2),

For (3) \quad $b^2 - 4ac = 0$

or $6^2 = 4 \times 2(1 + k)$

\[ 8 + 8k = 36 \Rightarrow 8k = 28 \]

\[ k = 3.5 \]

(iii) For $k = -9$

\[ f(x) = 2x - (-9) = 2x + 9 \quad \text{(4)}
\]

Now given $f(x) < g(x)$

From (3) \& (4)

\[ 2x^2 + 8x + 1 < 2x + 9 \]

or $2x^2 + 6x - 8 < 0$

\[ x^2 + 3x - 4 < 0 \]

\[ (x+4)(x-1) < 0 \]

\[ -4 < x < 1 \]

\[ \text{for} \quad (x-a)(x-b) < 0 \]

\[ b < x < a \]

\[ a > b \]

\[ b < x < a \]
Example 8: A line has equation \( y = 3kx - 2k \) and a curve has equation \( y = x^2 - kx + 2 \), where \( k \) is a constant.

(i) Find the set of values of \( k \) for which the line and curve meet at two distinct points.

(ii) For each of two particular values of \( k \), the line is a tangent to the curve. Show that these two tangents meet on the \( x \)-axis.

Solution: Line: \( y = 3kx - 2k \) \( \quad \boxed{1} \)

Curve: \( y = x^2 - kx + 2 \) \( \quad \boxed{2} \)

(i) For line intersects the curve from \( \boxed{1, 2} \)

\[ x^2 - kx + 2 = 3kx - 2k \]

or \( x^2 - 4kx + 2(2k + 2) = 0 \) \( \boxed{3} \)

for \( \boxed{1} \) and \( \boxed{2} \) intersect at two distinct points, for \( \boxed{3} \), \( b^2 - 4ac > 0 \)

\[ (4k)^2 - 4 \cdot 1 \cdot (2k + 2) > 0 \]

\[ 16k^2 - 8k - 8 > 0 \]

or \( 2k^2 - k - 1 > 0 \)

\( (2k + 1)(k - 1) > 0 \)

\[ \text{Roots of the eqn are } 1, -\frac{1}{2} \]

\( \boxed{2} \), \( \boxed{1} \)

\[ k < -\frac{1}{2} \text{ or } k > 1 \]

\( \checkmark \)

(ii) For line \( \boxed{1} \) is tangent to the curve \( \boxed{2} \)

for \( \boxed{3} \), \( b^2 - 4ac = 0 \)

\[ 16k^2 - 8k - 8 = 0 \]

\[ k = 1, -\frac{1}{2} \]

for \( \boxed{1} \), \( \boxed{2} \) not tangents

\( k = 1, \quad y = 3x - 2 \) \( \boxed{4} \)

\( k = -\frac{1}{2}, \quad y = -\frac{3}{2}x + 1 \) \( \boxed{5} \)

solve \( \boxed{4, 5} \) \( \Rightarrow x = \frac{3}{2} \)

\[ y = 0 \]

Tangents \( \boxed{4, 5} \) intersect at point \( \left( \frac{3}{2}, 0 \right) \), i.e. on \( x \)-axis.