Mathematics - A Level P_3

Complex Numbers **Exercise -1 [with Answers / Argand diagrams]**

1. The complex number (3- **i**) is denoted by u. Its conjugate is denoted by $\mathbf{u}^*$.
   (i) On argand diagram with origin O, Show the points A, B and C representing the complex numbers u, $\mathbf{u}^*$ and $\mathbf{u}^*$-u. respectively. What type of quadrilateral is OABC? **[4]**
   (ii) Show your working and without using calculator, express $\mathbf{u}^*$/u in the form x + **i**y, where x and y are real numbers. **[3]**
   (iii) By considering the argument of $\mathbf{u}^*$/ u ; prove that $\tan^{-1}\frac{3}{4} = 2\tan^{-1}\frac{1}{3}$. **W-15/ 31/ 32** **SP-3/ Q9/2017** **[3]**

2. (a) Find the complex no. z, satisfying the equation: $\mathbf{z}^*$+ 1 = 2**i**z, where $\mathbf{z}^*$denotes the complex conjugate of z, Give your answer in the form x + **i**y, where x and y are real numbers. **[5]**
   (b) (i) On a sketch of argand diagram, shade the region where points represent complex numbers satisfying the inequities.
   $|z+1-3i| \le 1$and Ima z ≥ 3, where ima z denotes the imaginary part of z. **March-16/ 32/ Q10** **[4]**
   (ii) Determine the difference between the greatest and least values of arg z for points lying in this region. **[2]**

3. (a) Solve the equation : **i** z^2 + 2z – 3**i** = 0, Give your answer in the form x + **i**y, where x and y are real numbers. **[5]**
   (b) (i) On a sketch of argand diagram, show the locus representing complex numbers satisfying the equation : $|z| = |z-4-3i|$ **[2]**
   (ii) Find the complex number represented by the point on the locus, where $|z|$ is least. Find the modulus and argument of this complex numbers giving the argument correct to two decimal places. **[3]** **S-16/ 32/ Q10**

4. (a) Showing all your working and without use of a calculator, find the square root of a complex numbers 7-6$\sqrt{2}$ **i.** Give your answer in the form x + **i**y, where x and y are real and exact. **[5]**
   (b) (i) On an argand diagram, sketch the loci of the points representing complex numbers w and z such that :
   $|w-1-2i| = 1$and arg (z-1) = $\frac{3}{4}\pi$ **[4]**
   S-16/ 31/ Q10
   (ii) Calculate the least value of $|w-z|$ for the points on these loci. **[2]**

5. Throughout this question the use of a calculator is **not permitted.** The complex numbers -1+3i and 2-i are denoted by u and v respectively. In an argand diagram with origin O, the points A, B and C respresent the numbers u, v and u + v respectively.
   (i) Sketch this diagram and state fully the geometrical relationship between OB and AC. **[4]**
   (ii) Find in the form x + **i**y, where x and y are real, the complex numbers u/v. **[3]**
   (iii) Prove that angle AOB = $\frac{3}{4}\pi$ **S-16/ 33/ Q9** **[2]**

6. Throughout this question the use of a calculator is **not permitted.**
   (a) Solve the equation: $(1+2i)w^2 + 4w - (1-2i) = 0$, giving your answer in the form x + **i**y, where x and y are real. **[5]**
   (b) On the sketch of an argand diagram, shade the region whose points represent complex numbers satisfying the inequalities: $|z-1-i| \le 2$ and $-\frac{1}{4}\pi \le \arg z \le \frac{1}{4}\pi$ **[5]** **W-16/31/ 32/ Q9**

7. Throughout this question the use of a calculator is **not permitted.**

The complex number z is defined by z = $\sqrt{2}$ - $\sqrt{6}$ **i**. The complex conjugate of z is denoted by $\mathbf{z}^*$.

(i) Find the modulus and argument of **z**. **[2]**

(ii) Express each of the following in the form x + **i**y, where x and y are real and exact.

(a) z + 2 $\mathbf{z}^*$.

(b) $\mathbf{z}^*$/ iz **[4]**

(iii) On a sketch of an Argand diagram with origin O, Show the points A and B representing the complex numbers $\mathbf{z}^*$ and iz respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$ **W-16/33/ Q7** **[3]**

8. The complex number w is defined by $w = \frac{22+4i}{(2-i)^2}$

(i) Without using a calculator, show that w = 2+4i **[3]**

(ii) It is given that p is a real number such $\frac{1}{4}\pi \leq \arg(w+p) \leq \frac{3}{4}\pi$. Find the set of possible values of p. **[3]**

(iii) The complex conjugate of w is denoted by w*. The complex numbers w and w* are represented in Argand diagram by the points S and T respectively. Find, in the form $|z-a| = k$, the equation of circle passing through S, T and the origin. **S-15/31/ Q8** **[3]**

9. The complex number u is given by u = -1 + 4$\sqrt{3}i$

(i) Without using a calculator and showing all your working. Find the two square roots of u. Give your answer in the form x + **i**y, where x and y are real and exact. **[5]**

(ii) On an Argan diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z-u| = 1$. Determine the greatest value of arg z for the points on this locus. **S-15/32/ Q7** **[4]**

10. The complex numbers 1-i is denoted by u.

(i) Showing your working and without using a calculator, express i/u; in the form x + **i**y, where x and y are real. **[2]**

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equation:

$|z-u| = |z|$ and $|z-i| = 2$ **[4]**

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part(ii) **S-15/33/ Q8** **[3]**

11. (a) It is given that (1 + 3i) w = 2 + 4i.

Showing all necessary working , prove that the exact value of $|w^2|$ is 2 and find arg (w^2) correct to 3 significant figures. **[6]**

(b) On a single argand diagram sketch the loci $|z| = 5$ and $|z-5| = |z|$. Hence determine the complex numbers represented by the points common to both loci, giving each answer in the form $re^{i\theta}$. **[4]**

W-15/33/ Q9

12. The complex number z is defined by z = $\frac{9\sqrt{3}+9i}{\sqrt{3}-i}$. Find showing all working

(i) An expression for z in the form $re^{i\theta}$, where r > 0 and $-\pi < \theta \leq \pi$ **[5]**

(ii) The two square roots of z, giving your answer in the form $re^{i\theta}$, where r > 0 and $-\pi < \theta \leq \pi$ **[3]**

S-14/31/ Q5

13. (a) It is given that -1+ $\sqrt{5}$ i is a root of the equation :

$z^3 + 2z + a = 0$, where a is real. Showing your working, find the value of a, and write down the other complex roots of this equation. **[4]**

(b) The complex numbers w has modulus 1 and argument 2θ radian. Show that $\frac{w-1}{w+1} = i\tan\theta$. **[4]**

S-14/32/ Q7

14. (a) The complex numbers $\frac{3-5i}{1+4i}$ is denoted by u. Showing your working, express u in the form (x+iy), where x and y are real. **[3]**

(b) (i) on a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities: $|z-2-i| \le 1$ and $|z-i| \le |z-2|$ **[4]**

(ii) Calculate the maximum value of arg z for points lying in the shaded region. **[2]**

S-14/33/ Q7

15. Throughout this question the use of a calculator is **not permitted.**

The complex numbers w and z satisfy the relation:

w = $\frac{z+i}{iz+2}$

(i) Given that z = 1+ i, find w, giving your answer in the form x + iy, where x and y are real. **[4]**

(ii) Given instead that w = z and the real part of z is negative, find z , giving your answer in the form x + iy, where x and y are real. **[4]**

W-14/31/32/ Q5

16. The complex numbers w and z are defined by : w = 5+3i and z = 4+ i.

(i) Express $\frac{iw}{z}$ in the form x + iy , showing all your working and giving the exact values of x and y. **[3]**

(ii) Find wz and hence, by considering arguments, Show that $\tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \frac{1}{4}\pi$ **[4]**

W-14/33/ Q5

17. (a)Without using a calculator, solve the equation:

$3w+2iw^* = 17 + 8i$,

Where w^* denotes the complex conjugate of w, give your answer in the form a+bi. **[4]**

(b) In an Argand diagram, the loci

arg(z – 2i) = $\frac{1}{6}\pi$ and $|z-3| = |z-3i|$

S-13/31/ Q7

intersect at the point P. Express the complex number represented by P in the form $re^{i\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. **[5]**

18. (a) The complex number w is such that Re w > 0 and $w+3w^*=iw^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form x+iy, where x and y are real. **[5]**

(b) On a sketch of an Argand diagram, Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z-2i| \le 2$ and $0 \le \arg(z+2) \le \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for the points in this region, giving your answer correct to 2 decimal places. **[6]**

S-13/32/ Q9

19. The complex numbers z = a + ib. The complex conjugate of z is denoted by z*.

(i) Show that $|z|^2 = z.z^*$ and that $(z-ki)^* = z^*+ki$, where k is real. In an Argand diagram a set of points representing complex numbers z is defined by the equation : $|z-10i| = 2|z-4i|$. **[2]**

(ii) Show by squaring both sides that : $zz^* - 2iz^* + 2iz - 12 = 0$. Hence show that $|z-2i| = 4$. **[5]**

(iii) Describe the set of points geometrically. . **S-13/33/ Q7** **[1]**

20. Throughout this question the use of a calculator is **not permitted.**
   (a) The complex numbers u and v satisfy the equation:
   u + 2v = 2i and iu + v = 3
   Solve the equation for u and v , giving both answer in the form x + iy, where x and y are real. **[5]**
   (b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z+i| = 1$ and the locus representing complex numbers w, satisfying arg(w-2) = $\frac{3}{4}\pi$. Find the least value of $|z-w|$ for the points on these loci. **W-13/31/32/ Q8** **[5]**

21. (a) Without using a calculator, use the formula for the solution of a quadratic equation to solve:
(2-i)z^2 + 2z + 2 + i = 0 give your answer in the form a+ib. **[5]**

(b) The complex number w is defined by w = $2e^{\frac{1}{4}\pi i}$. In an Argand diagram the points A, B and C represent the complex number w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. **[5]**

W-13/33/ Q9

22. The complex number u is defined by $u = \frac{(1+2i)^2}{(2+i)}$

(i) Without using a calculator and showing your working express u in the form x + iy, where x and y are real. **[4]**

(ii) Sketch an argand diagram showing the locus of the complex number z such that $|z-u| = |u|$ **[3]**

S-12/31/ Q4

23. The complex number u is defined by : $u = \frac{(1+2i)}{(1-3i)}$

(i) Express u in the form x + iy, where x and y are real. **[3]**

(ii) Show on a sketch of an Argand diagram the points A, B and C respectively the complex number u, 1 + 2i and 1 – 3i respectively. **[2]**

(iii) By considering the argument of 1 + 2i and 1 – 3i, Show that $\tan^{-1}2 + \tan^{-1}3 = \frac{3}{4}\pi$ **S-12/32/ Q4** **[3]**

24. (a) The complex numbers u and w satisfy the equation : u – w = 4i and uw = 5. Solve the equations for u and w, giving your answer in the form (x + iy), where x and y are real. **[5]**

(b) On a sketch of an argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z-2+2i| \le 2$ and $\text{Re} \ge 1$, where Re z denoted the real parts of z; arg z $\le -\frac{\pi}{4}$ **[5]**

(c) calculate the greatest possible value of Re z for points lying in the shaded region. **S-12/33/ Q10** **[1]**

25. The complex number $1+\sqrt{2}i$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by p(x).
(i) Showing your working, verify that u is a root of the equation p(x)=0, and write down a second complex root of the equation. **[4]**

(ii) Find the other two roots of the equation p(x) = 0. **W-12/31/32/Q9** **[6]**

26. (a) Without using a calculator, Solve the equation $iw^2 = (2-2i)^2$ **[3]**

(b) (i) Sketch an argand diagram showing the region R consisting of points representing the complex number z where $|z-4-4i| \le 2$ **[2]**

(ii) For complex numbers represented by the points in the region R, It is given that $p \le |z| \le q$ and $\alpha \le \arg z \le \beta$. Find the values of p, q, α and β, giving your answer correct to 3 significant figures. **[6]**

W-12/33/Q10

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Mathematics - A Level $\quad$ P_3

Complex Numbers **Answers**

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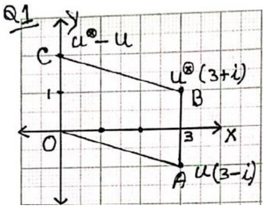
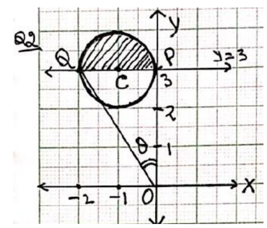
<tr>
<td>1.</td>
<td>(i) OABC is a parallelogram [$\because\ \vec{AB} = \vec{OC}$]


(ii) $\left(\frac{4}{5}+\frac{3}{5}i\right)$
(iii) use arg $\left(\frac{u^*}{u}\right)$ = arg u^* - arg u</td>
<td>2.</td>
<td>(a) z = $\left(\frac{1}{3}-\frac{2}{3}i\right)$


(b) (i) Shaded region
(ii) $\theta = \tan^{-1}\frac{2}{3} = 33.7^\circ$</td>
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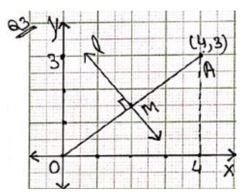
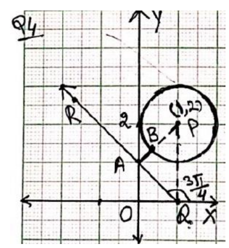
<tr>
<td>3.</td>
<td>(a) $(\sqrt{2}+i)$ and $(-\sqrt{2}+i)$
(b)(i)
Line *l* perpendicular bisector of OA is the required locus.


(ii) $|z|$ is least at M. [Mid-point of OA]
$|OM| = \frac{5}{2}$ and arg OM = $\tan^{-1}\frac{3}{4} = 36.87^\circ$</td>
<td>4.</td>
<td>(a) $\pm\left(3 - i\sqrt{2}\right)$
(b) (i)
Locus of w is circle with centre P(1,2), r = 1
Locus of z is $\vec{QR}$


(ii) Min $|w-z| = AB = AP - BP = (\sqrt{2}-1)$</td>
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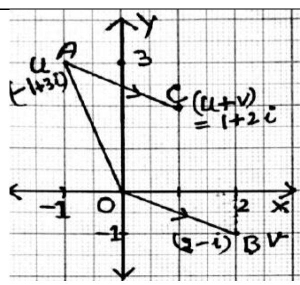
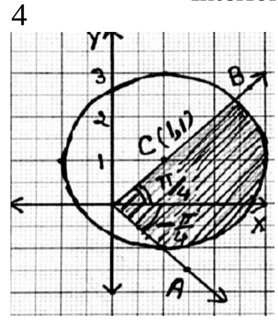
<td>5.</td>
<td>$\vec{OB} = 2 - i$
$\vec{AC} = (1+2i) - (-1+3i)$
$= (2-i)$


(i) $\vec{OB} = \vec{AC}$, OB and AC are parallel and equal.
(ii) $\frac{u}{v} = (-1+i)$
(iii) $\angle AOB = \arg u - \arg v = \arg\frac{u}{v} = \arg(-1+i)$
$= \tan^{-1} - 1 = \frac{3}{4}\pi$</td>
<td>6.</td>
<td>(a) (-1+2*i*) or $\left(\frac{1}{5}-\frac{2}{5}i\right)$
(b) $|z-(1+i)| \le 2$
Interior of circle centre C(1, 1) and radius = 2 and
$-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$ $\quad$ Interior of $\angle AOB$

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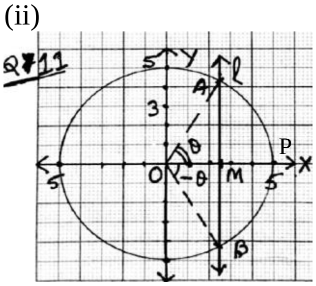
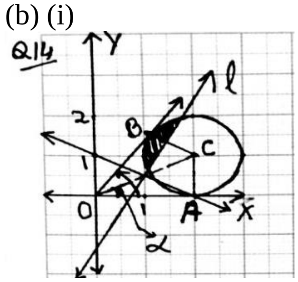
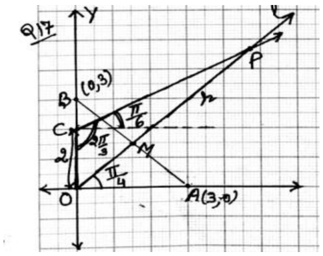
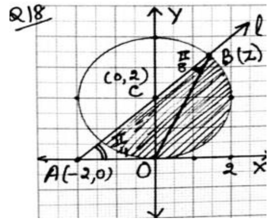
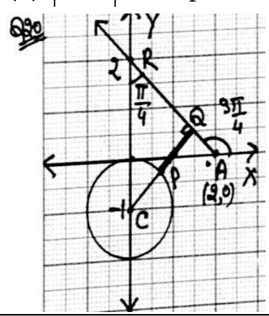
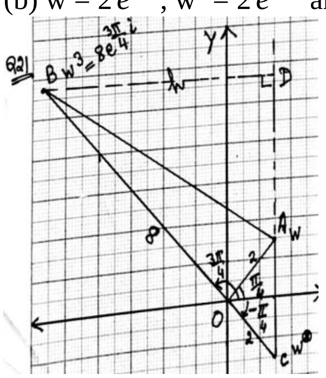
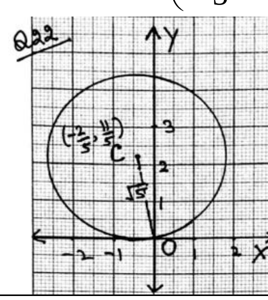
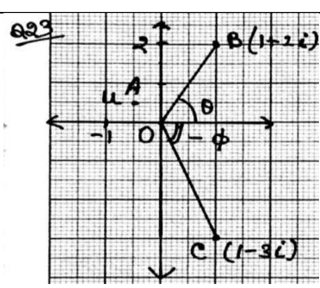
7.	(i) $\lvert z\rvert = 2\sqrt{2}$; arg z = $\frac{-\pi}{3}$ (ii) (a) z + 2z* = $\left(3\sqrt{2}+\sqrt{6}i\right)$ (b) $\frac{z^*}{iz} = \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ (iii) z* = $\sqrt{2}+\sqrt{6}i \to A$ iz = $\left(\sqrt{6}+\sqrt{2}i\right) \to B$ $\angle AOB = \arg z^* - \arg(iz) = \arg\frac{z}{iz} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$	8.	(i) 2 + 4i = w (ii) $-6 \le p \le 2$ (iii) $\lvert z-5\rvert = 5$
9.	(i) $\pm\left(\sqrt{3}+2i\right)$ (ii) arg z = 106.4° or (1.86 radian)	10.	(i) $\left(-\frac{1}{2}+\frac{1}{2}i\right)$ (ii) Perpendicular Bisector of segment joining O(0,0) and u(1,-1). And Circle C (0,1) and r = 2 (iii) Point of intersection (2 + i) $\Rightarrow \arg = 26.6^\circ$ and (0,-1) and arg = $-\frac{\pi}{2}$
11.	Find w = $\left(\frac{7}{5}-\frac{1}{5}i\right)$ $\therefore w^2 = \left(\frac{48}{25}-\frac{14}{25}i\right)$ (i) $\lvert w^2\rvert = \sqrt{\left(\frac{48}{25}\right)^2+\left(-\frac{14}{25}\right)^2} = 2$ arg w² = $\tan^{-1}\left(-\frac{14}{48}\right) = -0.284$ or (-16.3°) (ii)  (i) $\lvert z\rvert = 5$ circle centre at O. and r = 5 (ii) $\lvert z-5\rvert = \lvert z\rvert$ is perpendicular bisector of OP intersect at A and B. OA = r, $\theta = \tan^{-1}\left(\frac{5\sqrt{3}/2}{5/2}\right)$ $= \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ Complex Number Ans $\therefore$ Point A and B $= 5\cdot e^{\pm\frac{\pi}{3}i}$	12.	(i) z = $9e^{i\frac{\pi}{3}}$ (ii) $3e^{\frac{\pi}{6}i}$ and $3e^{\frac{-5\pi}{6}i}$
		13.	(i) a = -12 Second complex root is $\left(-1-\sqrt{5}i\right)$ and 2 (ii) $\lvert w\rvert = 1$, arg w = 2θ $\therefore$ w = ($\cos 2\theta + i\sin 2\theta$) L.H.S. $\frac{w-1}{w+1} = \frac{(\cos 2\theta + i\sin 2\theta)-1}{(\cos 2\theta + i\sin 2\theta)+1}$ and proceed.
		14.	(a) (-1-i) (b) (i)  $\lvert z-(2+i) \le 1\rvert$ Circle C(2,1) and radius = 1 b(ii) Reg $\angle BOA = 2\alpha$ $\alpha = \tan^{-1}\frac{1}{2} = 26.56^\circ$ $\therefore \angle BOA = 2\times 26.56$ $= 53.1^\circ$ $\lvert z-i\rvert \le \lvert z-2\rvert$ Half plane of perpendicular bisector of (0,1) and (2,0)
15.	(i) $\left(\frac{3}{2}+\frac{1}{2}i\right)$ (ii) $\left(-\frac{3}{2}+\frac{1}{2}i\right)$	16.	(i) $\left(-\frac{7}{17}+\frac{23}{17}i\right)$ (ii) wz = (17 + 17i) Use arg wz = arg w + arg z; $\frac{\pi}{4} = \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}$

17.	(a) w = (7 – 2i) (b) $\|z-3\| = \|z-3i\|$ perpendicular bisector of A(3,0), B(0,3) $\rightarrow OP$, arg (z-2i) = $\frac{\pi}{6}$  Sine Rule $\frac{r}{\sin 2\pi} = \frac{2}{\sin\frac{\pi}{12}}$ $\Rightarrow r = 6.69$ $P(z) = re^{i\theta} = 6.69e^{\frac{\pi}{4}i}$	18.	(a) w = $\left(2\sqrt{2}-2i\right)$ (b) $\|z-2i\| \le 2$ is a circle C(0,2), r = 2  And $0 \le \arg\left(z-(-2)\right) \le \frac{\pi}{4}$ Half line AB, A(-2,0), arg $\frac{\pi}{4}$, OB = $\|z\| = 3.7$
19.	(iii)Represent a circle with Centre at 2i and radius = 4	20.	(a) u = (-2-2i) and v = (1+2i) (b) $\|z+i\| = 1$ represents a circle. C(0,-1) and r = 1 and  arg(w-2) = $\frac{3\pi}{4}$ is a half line through A(2,0) and an angle $\frac{3\pi}{4}$. Min $\|z-w\| = PQ = CQ - CP$ $= \left(\frac{3}{\sqrt{2}}-1\right)$
21.	(a) $\left(-\frac{4}{5}+\frac{3}{5}i\right)$ and –i (b) w = $2e^{\frac{\pi}{4}i}$, w* = $2e^{\frac{-\pi}{4}i}$ and w^{3} = 8. $e^{\frac{3\pi}{4}i}$  Area $\Delta ABC = \frac{1}{2}\times AC \times BD$ $= \frac{1}{2}\times 2\sqrt{2}\times\frac{10}{\sqrt{2}}$ $= 10$	22.	(i) u = $\left(-\frac{2}{5}+\frac{11}{5}i\right)$ (ii) $\|z-u\| = \|u\|$; $\left\|z-\left(\frac{-2}{5}+\frac{11}{5}i\right)\right\| = \sqrt{5}$ Circle Centre $\left(-\frac{2}{5}+\frac{11}{5}i\right)$ and r = $\sqrt{5}$ 
23.	 (i) u = $\left(-\frac{1}{2}+\frac{1}{2}i\right)$ (ii) arg (1+2i) = $\theta = \tan^{-1} 2$ arg(1-3i)= $\phi = \tan^{-1} 3$ $\angle COB = \theta + \phi$	23.	(iii) Now $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{2+3}{1-2\times 3} = -1$ $\theta + \phi = \tan^{-1}(-1)$ $\Rightarrow \tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$

<table>
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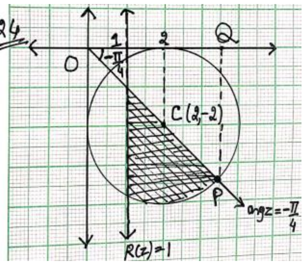
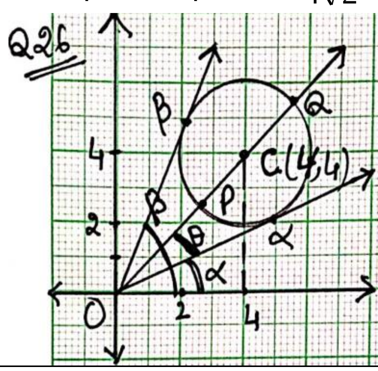
<td>24.</td>
<td>
(i) $\begin{cases} u = 1 + 2i \\ w = 1 - 2i \end{cases}$ or $\begin{cases} u = -1 + 2i \\ w = -1 - 2i \end{cases}$


Greatest value of Re z is at P = OQ
Now OP = OC + CP
= $2\sqrt{2} + 2$
OQ = OP . cos 45° = $\frac{OP}{\sqrt{2}}$
= $\frac{2\sqrt{2}+2}{\sqrt{2}} = 2 + \sqrt{2}$
</td>
<td>25.</td>
<td>
(i) $\left(1 - \sqrt{2}i\right)$
(ii) (-1-i) , (-1+ i)
</td>
</tr>
<tr>
<td>26.</td>
<td>
(a) w = $\pm 2\sqrt{2}i$
(b)(i) $\left|z - (4 + 4i)\right| \le 2$ Circle C(4+4i) and r = 2
(ii) $p \le |z| \le q$
p = OC – CP = $4\sqrt{2} - 2 = 3.66$
q = $4\sqrt{2} + 2 = 7.66$
$\alpha = \frac{\pi}{4} - \theta = \frac{\pi}{4} - \sin^{-1}\frac{2}{4\sqrt{2}} = 0.424$ radian
$\beta = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \sin^{-1}\frac{2}{4\sqrt{2}} = 1.15$ radian


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