1. The complex number (3 - i) is denoted by u. Its conjugate is denoted by \( u^* \).
   (i) On an argand diagram with origin O, show the points A, B and C representing the complex numbers u, \( u^* \) and \( u^-u \), respectively. What type of quadrilateral is OABC? [4]
   (ii) Show your working and without using calculator, express \( u^*/u \) in the form \( x + iy \), where \( x \) and \( y \) are real numbers. [3]
   (iii) By considering the argument of \( u^*/u \), prove that \( \tan^{-1} \frac{3}{4} = 2 \tan^{-1} \frac{1}{3} \). [3]

2. (a) Find the complex no. \( z \), satisfying the equation: \( z^2 + 1 = 2iz \), where \( z^* \) denotes the complex conjugate of \( z \). Give your answer in the form \( x + iy \), where \( x \) and \( y \) are real numbers. [5]
   (b) (i) On a sketch of an argand diagram, shade the region where points represent complex numbers satisfying the inequities:
       \[ |z + 1 - 3i| \leq 1 \text{ and } \text{Im}z \geq 3 \], where \( \text{Im}z \) denotes the imaginary part of \( z \). [4]
       (ii) Determine the difference between the greatest and least values of \( \arg z \) for points lying in this region. [2]

3. (a) Solve the equation: \( i z^2 + 2z - 3i = 0 \), Give your answer in the form \( x + iy \), where \( x \) and \( y \) are real numbers. [5]
   (b) (i) On a sketch of an argand diagram, show the locus representing complex numbers satisfying the equation: \( |z| = |z - 4 - 3i| \) [2]
       (ii) Find the complex number represented by the point on the locus, where \( |z| \) is least. Find the modulus and argument of this complex numbers giving the argument correct to two decimal places. [3]

4. (a) Show all your working and without use of a calculator, find the square root of a complex numbers \( 7 - 6\sqrt{2} \text i \). Give your answer in the form \( x + iy \), where \( x \) and \( y \) are real and exact. [5]
   (b) (i) On an argand diagram, sketch the loci of the points representing complex numbers \( w \) and \( z \) such that:
       \[ |w - 1 - 2i| = 1 \text{ and } \arg (z-1) = \frac{3}{4} \pi \] [4]
       (ii) Calculate the least value of \( |w - z| \) for the points on these loci. [2]

5. Throughout this question the use of a calculator is not permitted. The complex numbers \(-1+3i \) and \(2-i \) are denoted by \( u \) and \( v \) respectively. In an argand diagram with origin O, the points A, B and C represent the numbers \( u \), \( v \) and \( u + v \) respectively.
   (i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]
   (ii) Find in the form \( x + iy \), where \( x \) and \( y \) are real, the complex numbers \( u/v \). [3]
   (iii) Prove that angle AOB = \( \frac{3}{4} \pi \) \( S-16/33/Q9 \) [2]

6. Throughout this question the use of a calculator is not permitted.
   (a) Solve the equation: \((1 + 2i)w^2 + 4w - (1 - 2i) = 0 \), giving your answer in the form \( x + iy \), where \( x \) and \( y \) are real. [5]
   (b) On the sketch of an argand diagram, shade the region whose points represent complex numbers satisfying the inequalities:
       \[ |z - 1 - i| \leq 2 \text{ and } -\frac{1}{4} \pi \leq \arg z \leq \frac{1}{4} \pi \] \( W-16/31/32/Q9 \) [5]
7. Throughout this question the use of a calculator is **not permitted**.

The complex number \( z \) is defined by \( z = \sqrt{2} - \sqrt{6} \, i \). The complex conjugate of \( z \) is denoted by \( z^* \).

(i) Find the modulus and argument of \( z \). [2]

(ii) Express each of the following in the form \( x + iy \), where \( x \) and \( y \) are real and exact.

(a) \( z + 2 \, z^* \).

(b) \( z^* / iz \). [4]

(iii) On a sketch of an Argand diagram with origin \( O \), show the points A and B representing the complex numbers \( z^* \) and \( iz \) respectively. Prove that angle \( AOB \) is equal to \( \frac{1}{6} \pi \) W-16/33/ Q7 [3]

8. The complex number \( w \) is defined by \( w = \frac{22 + 4i}{(2-i)^2} \).

(i) Without using a calculator, show that \( w = 2+4i \). [3]

(ii) It is given that \( p \) is a real number such \( \frac{1}{4} \pi \leq \text{arg}(w + p) \leq \frac{3}{4} \pi \). Find the set of possible values of \( p \). [3]

(iii) The complex conjugate of \( w \) is denoted by \( w^* \). The complex numbers \( w \) and \( w^* \) are represented in Argand diagram by the points S and T respectively. Find, in the form \( |z - d| = k \), the equation of circle passing through S, T and the origin. [3]

9. The complex number \( u \) is given by \( u = -1 + 4 \sqrt{3}i \).

(i) Without using a calculator and showing all your working, find the two square roots of \( u \). Give your answer in the form \( x + iy \), where \( x \) and \( y \) are real and exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers \( z \) satisfying the relation \( |z - u| = 1 \). Determine the greatest value of \( \text{arg} \, z \) for the points on this locus. S-15/31/ Q8 [3]

10. The complex numbers 1-i is denoted by \( u \).

(i) Showing your working and without using a calculator, express \( i/u \); in the form \( x + iy \), where \( x \) and \( y \) are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers \( z \) satisfying the equation: \( |z - u| = \frac{1}{2} \) and \( |z - i| = 2 \). [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part(ii) S-15/33/ Q8 [3]

11. (a) It is given that \((1 + 3i) \, w = 2 + 4i\).

Showing all necessary working, prove that the exact value of \( |w^2| \) is 2 and find \( \text{arg} \, (w^2) \) correct to 3 significant figures. [6]

(b) On a single argand diagram sketch the loci \( |z| = 5 \) and \( |z - 5| = |z| \). Hence determine the complex numbers represented by the points common to both loci, giving each answer in the form \( re^{i\theta} \). W-15/33/ Q9 [4]

12. The complex number \( z \) is defined by \( z = \frac{9\sqrt{3} + 9i}{\sqrt{3} - i} \). Find showing all working

(i) An expression for \( z \) in the form \( re^{i\theta} \), where \( r > 0 \) and \( -\pi < \theta \leq \pi \). [5]

(ii) The two square roots of \( z \), giving your answer in the form \( re^{i\theta} \), where \( r > 0 \) and \( -\pi < \theta \leq \pi \). S-14/31/ Q5 [3]
13. (a) It is given that \(-1+ \sqrt{5} \) \(i\) is a root of the equation:
\[ z^3 + 2z + a = 0 , \text{ where } a \text{ is real.} \]
Showing your working, find the value of \(a\), and write down the other complex roots of this equation. [4]

(b) The complex numbers \(w\) has modulus 1 and argument 2 \(\theta\) radian. Show that \[ \frac{w - 1}{w + 1} = i \tan \theta. \] [4]

14. (a) The complex numbers \[ \frac{3 - 5i}{1 + 4i} \] is denoted by \(u\). Showing your working, express \(u\) in the form \((x + iy)\), where \(x\) and \(y\) are real. [3]

(b) (i) on a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities: \[ |z - 2 - i| \leq 1 \text{ and } |z - i| \leq |z - 2| \] [4]

(ii) Calculate the maximum value of \(\text{arg } z\) for points lying in the shaded region. [2]

15. Throughout this question the use of a calculator is not permitted.

The complex numbers \(w\) and \(z\) satisfy the relation:

\[ w = \frac{z + i}{iz + 2} \]

(i) Given that \(z = 1 + i\), find \(w\), giving your answer in the form \(x + iy\), where \(x\) and \(y\) are real. [4]

(ii) Given instead that \(w = z\) and the real part of \(z\) is negative, find \(z\), giving your answer in the form \(x + iy\), where \(x\) and \(y\) are real. [4]

16. The complex numbers \(w\) and \(z\) are defined by: \(w = 5 + 3i\) and \(z = 4 + i\).

(i) Express \[ \frac{iw}{z} \] in the form \(x + iy\), showing all your working and giving the exact values of \(x\) and \(y\). [3]

(ii) Find \(wz\) and hence, by considering arguments, Show that \[ \tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \frac{1}{4}\pi \] [4]

17. (a) Without using a calculator, solve the equation:
\[ 3w + 2iw^* = 17 + 8i, \]
Where \(w^*\) denotes the complex conjugate of \(w\), give your answer in the form \(a + bi\). [4]

(b) In an Argand diagram, the loci
\[ \text{arg}(z - 2i) = \frac{1}{6}\pi \text{ and } |z - 3| = |z - 3i| \]
interact at the point \(P\). Express the complex number represented by \(P\) in the form \(re^{i\theta}\), giving the exact value of \(\theta\) and the value of \(r\) correct to 3 significant figures. [5]

18. (a) The complex number \(w\) is such that \(\text{Re } w > 0\) and \(w + 3w^* = iw^2\), where \(w^*\) denotes the complex conjugate of \(w\). Find \(w\), giving your answer in the form \(x + iy\), where \(x\) and \(y\) are real. [5]

(b) On a sketch of an Argand diagram, Shade the region whose points represent the complex numbers \(z\) which satisfy both the inequalities \[ |z - 2i| \leq 2 \text{ and } 0 \leq \text{arg}(z + 2) \leq \frac{1}{4}\pi. \] Calculate the greatest value of \(|z|\) for the points in this region, giving your answer correct to 2 decimal places. [6]
19. The complex numbers \( z = a + ib \). The complex conjugate of \( z \) is denoted by \( z^* \).

(i) Show that \( |z|^2 = z \cdot z^* \) and that \( (z-ki) = z^* + ki \), where \( k \) is real. In an Argand diagram a set of points representing complex numbers \( z \) is defined by the equation: \( |z - 10i| = 2|z - 4i| \). \([2]\)

(ii) Show by squaring both sides that: \( zz^* - 2iz^* + 2iz - 12 = 0 \). Hence show that \( |z - 2i| = 4 \). \([5]\)

(iii) Describe the set of points geometrically. \([1]\)

20. Throughout this question the use of a calculator is not permitted.

(a) The complex numbers \( u \) and \( v \) satisfy the equation: \( u + 2v = 2i \) and \( iv + v = 3 \).

Solve the equation for \( u \) and \( v \), giving both answers in the form \( x + iy \), where \( x \) and \( y \) are real. \([5]\)

(b) On an Argand diagram, sketch the locus representing complex numbers \( z \) satisfying \( |z + i| = 1 \) and the locus representing complex numbers \( w \), satisfying \( \text{arg}(w-2) = \frac{3\pi}{4} \). Find the least value of \( |z-w| \) for the points on these loci. \([5]\)

21. (a) Without using a calculator, use the formula for the solution of a quadratic equation to solve: \((2-i)z^2 + 2z + 2 + i = 0 \) give your answer in the form \( x + iy \), where \( x \) and \( y \) are real. \([5]\)

(b) The complex number \( w \) is defined by \( w = 2e^{\frac{3\pi i}{4}} \). In an Argand diagram the points A, B and C represent the complex number \( w \), \( w^3 \) and \( w^* \) respectively (where \( w^* \) denotes the complex conjugate of \( w \)). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. \([5]\)

22. The complex number \( u \) is defined by \( u = \frac{(1 + 2i)^2}{(2 + i)} \)

(i) Without using a calculator and showing your working express \( u \) in the form \( x + iy \), where \( x \) and \( y \) are real. \([4]\)

(ii) Sketch an Argand diagram showing the locus of the complex number \( z \) such that \( |z - u| = |u| \). \([3]\)

23. The complex number \( u \) is defined by \( u = \frac{(1 + 2i)}{(1 - 3i)} \)

(i) Express \( u \) in the form \( x + iy \), where \( x \) and \( y \) are real. \([3]\)

(ii) Show on a sketch of an Argand diagram the points A, B and C respectively the complex number \( u \), \( 1 + 2i \) and \( 1 - 3i \) respectively. \([2]\)

(iii) By considering the argument of \( 1 + 2i \) and \( 1 - 3i \), Show that \( \tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4} \). \([3]\)

24. (a) The complex numbers \( u \) and \( w \) satisfy the equation: \( u - w = 4i \) and \( uw = 5 \). Solve the equations for \( u \) and \( w \), giving your answer in the form \( (x + iy) \), where \( x \) and \( y \) are real. \([5]\)

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities \( |z - 2 + 2i| \leq 2 \) and \( \text{Re} \geq 1 \), where \( \text{Re} \) \( z \) denoted the real parts of \( z \); \( \text{arg} z \leq -\frac{\pi}{4} \). \([5]\)

(c) calculate the greatest possible value of \( \text{Re} \) \( z \) for points lying in the shaded region. \([1]\)
25. The complex number $1 + \sqrt{2}i$ is denoted by $u$. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by $p(x)$.

(i) Showing your working, verify that $u$ is a root of the equation $p(x) = 0$, and write down a second complex root of the equation.

(ii) Find the other two roots of the equation $p(x) = 0$.

26. (a) Without using a calculator, Solve the equation $iw^2 = (2-2i)^2$

(b) (i) Sketch an argand diagram showing the region $R$ consisting of points representing the complex number $z$ where $|z - 4 - 4i| \leq 2$

(ii) For complex numbers represented by the points in the region $R$, it is given that $p \leq |z| \leq q$ and $\alpha \leq \arg z \leq \beta$. Find the values of $p$, $q$, $\alpha$ and $\beta$, giving your answer correct to 3 significant figures.
## Mathematics - A Level

### P3

#### Complex Numbers

### Answers

1. (i) OABC is a parallelogram \[ \because \overrightarrow{AB} = \overrightarrow{OC} \]

(ii) \[ \left( \frac{4}{5} + \frac{3}{5}i \right) \]

(iii) \[ \text{use } \arg \left( \frac{u^*}{u} \right) = \text{arg } u^* - \text{arg } u \]

2. (a) \[ z = \left( \frac{1}{3} - \frac{2}{3}i \right) \]

(b) (i) Shaded region

(ii) \[ \theta = \tan^{-1} \frac{2}{3} = 33.7^\circ \]

3. (a) \( (\sqrt{2} + i) \) and \( (−\sqrt{2} + i) \)

(b)(i) Line \( l \) perpendicular bisector of \( OA \) is the required locus.

(ii) \[ |z| \text{ is least at } M. \text{ [Mid-point of } OA] \]

\[ |OM| = \frac{5}{2} \text{ and } \arg OM = \tan^{-1} \frac{3}{4} = 36.87^\circ \]

4. (a) \( ±\left(3 - i\sqrt{2}\right) \)

(b) (i) Locus of \( w \) is circle with centre \( P(1,2) \), \( r = 1 \)

Locus of \( z \) is \( \overrightarrow{QR} \)

(ii) \[ \text{Min } |w - z| = AB = AP - BP = (\sqrt{2} - 1) \]

5. \[ \overrightarrow{OB} = 2 - i \]

\[ \overrightarrow{AC} = (1 + 2i) - (−1 + 3i) \]

\[ = (2 - i) \]

(i) \[ \overrightarrow{OB} = \overrightarrow{AC} , \text{ OB and } AC \text{ are parallel and equal.} \]

(ii) \[ \frac{u}{v} = (−1 + i) \]

(iii) \[ \angle AOB = \arg u - \arg v = \arg \frac{u}{v} = \arg(−1 + i) \]

\[ = \tan^{-1} - 1 = \frac{3}{4} \pi \]

6. (a) \( (−1+2i) \) or \( \left( \frac{1}{5} - \frac{2}{5}i \right) \)

(b) \[ |z - (1 + i)| \leq 2 \]

Interior of circle centre \( C(1, 1) \) and radius = 2 and \[ -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4} \]

Interior of \( \angle AOB \)
7. (i) \(|z| = 2\sqrt{2}\); \(\arg z = \frac{-\pi}{3}\)
(ii) (a) \(z + 2z^* = (3\sqrt{2} + \sqrt{6}i)\)
(b) \(\frac{z^*}{iz} = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\)
(iii) \(z^* = \sqrt{2} + \sqrt{6}i\rightarrow A\)
\(iz = \left(\sqrt{6} + \sqrt{2}i\right)\rightarrow B\)
\(\angle AOB = \arg z^* - \arg(iz) = \arg \frac{z}{iz} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}\)

8. (i) \(2 + 4i = w\)
(ii) \(-6 \leq p \leq 2\)
(iii) \(|z - 5| = 5\)

9. (i) \(\pm (\sqrt{3} + 2i)\)
(ii) \(\arg z = 106.4^\circ\) or (1.86 radian)

10. (i) \(\left(-\frac{1}{2} + \frac{1}{2}i\right)\)
(ii) Perpendicular Bisector of segment joining O(0,0) and u(1,-1).
\(\text{And Circle C (0,1) and r = 2}\)
(iii) Point of intersection \((2 + i) \Rightarrow \arg = 26.6^\circ\) and \((0,-1)\) and \(\arg = \frac{-\pi}{2}\)

11. Find \(w = \left(\frac{7}{5} - \frac{1}{5}i\right)\)
\(\therefore w^2 = \left(\frac{48}{25} - \frac{14}{25}i\right)\)
(i) \(|w|^2 = \sqrt{\left(\frac{48}{25}\right)^2 + \left(-\frac{14}{25}\right)^2} = 2\)
\(\arg w^2 = \tan^{-1}\left(-\frac{14}{48}\right) = -0.284\) or \((-16.3^\circ)\)
(ii) \(\frac{w}{\overline{w}} = \left(\frac{17 + 17i}{17 - 17i}\right)\)

12. (i) \(z = 9e^{\frac{i\pi}{3}}\)
(ii) \(3e^{\frac{i\pi}{6}}\) and \(3e^{\frac{-i\pi}{6}}\)

13. (i) \(a = -12\)
Second complex root is \((-1 - \sqrt{5}i)\) and 2
(ii) \(|w| = 1\), \(\arg w = 2\theta\)
\(\therefore w = (\cos 2\theta + i \sin 2\theta)\)
L.H.S. \(\frac{w - 1}{w + 1} = \frac{(\cos 2\theta + i \sin 2\theta - 1)}{(\cos 2\theta + i \sin 2\theta + 1)}\) and proceed.

14. (a) \((-1-i)\)
(b) (i) \(|z - (2 + i) \leq |\) Circle C(2,1) and radius = 1
(b(ii) Reg \(\angle BOA = 2\alpha\)
\(\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26.56^\circ\)
\(\therefore \angle BOA = 2 \times 26.56 = 53.1^\circ\)

15. (i) \(\left(\frac{3}{2} + \frac{1}{2}i\right)\) (ii) \(\left(\frac{-3}{2} + \frac{1}{2}i\right)\)

16. (i) \(\left(-\frac{7}{17} + \frac{23}{17}i\right)\) (ii) \(wz = (17 + 17i)\)
Use \(\arg wz = \arg w + \arg z;\) \(\frac{\pi}{4} = \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\)
17. (a) $w = (7 - 2i)$  
(b) $|z - 3| = |z - 3i|$ perpendicular bisector of A(3,0), B(0,3) → OP, arg $(z-2i) = \frac{\pi}{6}$  

\[ P(z) = re^{i\theta} = 6.69e^{\frac{\pi}{4}} \]

18. (a) $w = \left(2\sqrt{2} - 2i\right)$  
(b) $|z - 2i| \leq 2$ is a circle C(0,2), $r = 2$  

And $0 \leq \arg(z - (-2)) \leq \frac{\pi}{4}$  

Half line AB, A(-2,0), arg $\frac{\pi}{4}$, OB = $|z| = 3.7$

19. (iii) Represent a circle with Centre at 2i and radius = 4

20. (a) $u = (-2.2i)$ and $v = (1+2i)$  
(b) $|z + i| \leq 1$ represents a circle. C(0,-1) and $r = 1$ and $\arg(u-2) = \frac{3\pi}{4}$ is a half line through A(2,0) and an angle $\frac{3\pi}{4}$

\[ \text{Min} |z - w| = PQ = CQ - CP = \left(\frac{3}{\sqrt{2}} - 1\right) \]

21. (a) $\left(-\frac{4}{5} + \frac{3}{5}i\right)$ and $-i$

(b) $w = 2e^{\frac{\pi}{4}}, w^* = 2e^{-\frac{\pi}{4}}$ and $w^3 = 8. e^{\frac{3\pi}{4}}$

Area $\Delta ABC = \frac{1}{2} \times AC \times BD = \frac{1}{2} \times 2\sqrt{2} \times \frac{10}{\sqrt{2}} = 10$

22. (i) $u = \left(-\frac{2}{5} + \frac{11}{5}i\right)$

(ii) $|z - u| = |u|; \quad |z - \left(-\frac{2}{5} + \frac{11}{5}i\right)| = \sqrt{5}$

Circle Centre $\left(-\frac{2}{5} + \frac{11}{5}i\right)$ and $r = \sqrt{5}$

23. (iii) Now  
\[ \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{2 + 3}{1 - 2 \times 3} = -1 \]
 \[ \theta + \phi = \tan^{-1}(-1) = \tan^{-1}(3) \]
 \[ \Rightarrow \tan^{-1}(2) + \tan^{-1}(3) = \frac{3\pi}{4} \]
24. (i) \[ u = 1 + 2i \quad \text{or} \quad w = 1 - 2i \]
(ii) \[ u = -1 + 2i \quad \text{or} \quad w = -1 - 2i \]

Greatest value of \( \text{Re} \, z \) is at \( P = OQ \)
Now \( OP = OC + CP \)
\[ = 2\sqrt{2} + 2 \]
\[ OQ = OP \cdot \cos 45^\circ = \frac{OP}{\sqrt{2}} \]
\[ = \frac{2\sqrt{2} + 2}{\sqrt{2}} = 2 + \sqrt{2} \]

25. (i) \( 1 - \sqrt{2}i \)
(ii) (-1-i), (-1+i)

26. (a) \( w = \pm 2\sqrt{2}i \)
(b)(i) \( |z - (4 + 4i)| \leq 2 \) Circle C(4+4i) and \( r = 2 \)
(ii) \( p \leq |z| \leq q \)
\[ p = OC - CP = 4\sqrt{2} - 2 = 3.66 \]
\[ q = 4\sqrt{2} + 2 = 7.66 \]
\[ \alpha = \frac{\pi}{4} - \theta = \frac{\pi}{4} - \sin^{-1} \frac{2}{4\sqrt{2}} = 0.424 \text{ radian} \]
\[ \beta = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \sin^{-1} \frac{2}{4\sqrt{2}} = 1.15 \text{ radian} \]