A- LEVEL – MATHEMATICS P_3 Complex Numbers

(NOTES)

1. Given a **quadratic equation** :

$x^2 + 1 = 0$ or ($x^2 = -1$) has no solution in the set of real numbers, as there does not exist any real number whose square is -1.

Here we introduce a number **(symbol) $i = \sqrt{-1}$**

or $i^2 = -1$

and we may deduce $i^3 = -i$

$i^4 = 1$

Now solution of $x^2 + 1 = 0$

$\Rightarrow x^2 = -1$

$x = \pm \sqrt{-1}$

or $x = \pm i$

2. We define a complex number $z = (x + i y)$; $x, y \in R$

Example : $(4 + 3i)$, $\frac{5}{3}$, $7i$ and 0 are complex numbers.

a) Given a complex number $z = (a + i b)$

Then real part of z = a or Re z = a

and Imaginary part of z = b or img z = b

b) **Example** i) $z = (4 + 3i)$ is a complex number

ii) $\frac{5}{3} = \left(\frac{5}{3} + 0i\right)$ is pure real number

iii) $7i = (0 + 7i)$ is pure imaginary number

and $0 = 0 + i0$

3. a) Equal Complex numbers

Given Complex numbers $u = (a + i\,b)$ and $v = (c + i\,d)$

Then $(a + i\,b) = (c + i\,d) \Longleftrightarrow \begin{cases} a = c \\ b = d \end{cases}$ real and imaginary parts are separately equal

b) **Inequality** in complex numbers is **not** defined

$u = (a + i\,b)$, $v = (c + i\,d)$

$u > v$ or $v > u$ is **not defined**

4. Algebra of Complex Numbers

a) <u>Addition of Complex Numbers</u> :

Given $\mathbf{u} = (a + i\,b)$ and $\mathbf{v} = (c + i\,d)$; a, b, c and $d \in R$

$\mathbf{u + v} = ((a + c) + i\,(b + d))$

Example : $(2 + 3i) + (4 - 7i) = (2 + 4) + i\,(3 + (-7))$

$= (6 - 4i)$

b) <u>Subtraction of Complex Numbers</u> :

$\mathbf{u - v} = (a + i\,b) - (c + i\,d)$

$= (a - c) + i\,(c - d)$

Example : $(2 + 3i) - (4 - 7i) = (2 - 4) + i\,(3 - (-7))$

$= (-2 + 10i)$

c) <u>Multiplication of two complex numbers</u> :

$(a + i\,b)(c + i\,d) = (ac - cd) + i\,(ad + bc)$

Example : $(2 + 3i)(4 - 7i) = (2 \times 4 - 3(-7)) + i\,(2(-7) + 3 \times 4)$

$= (8 + 21) + i\,(-14 + 12)$

$= (29 - 2i)$

Note : i) $(a + i b)^2 = (a^2 - b^2) + 2abi$ — $(2 + 3i)^2 = (4 - 9) + 2 \times 2 \times 3i = (-5 + 12i)$

ii) $(a + i b)(a - i b) = a^2 + b^2$ — $(2 + 3i)(2 - 3i) = 2^2 + 3^2 = 13$

iii) $(a - i b)^2 = (a^2 - b^2) - 2abi$ — $(2 - 3i)^2 = (4 - 9) - 2 \times 2 \times 3 = (-5 - 12i)$

d) Division of two Complex Numbers :

$$\frac{(a+ib)}{(c+id)} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)} = \frac{(ac+bd)+i(bc-ad)}{(c^2+d^2)}$$

$$\frac{(2+3i)}{(4-7i)} = \frac{(2+3i)}{(4-7i)} \times \frac{(4+7i)}{(4+7i)} = \frac{(8-21)+i(14+12)}{(4^2+7^2)} = \frac{-13+26i}{16+49} = \frac{-13}{65} + \frac{26}{65} i = \left(-\frac{1}{5} + \frac{2}{5} i\right)$$

5. Conjugate of a Complex Numbers :

Given $z = (x + iy)$ $\quad x, y \in R$

Conjugate of z denoted by $z^* = (x - yi)$

Example : i) $z = 4 + 3i \Longrightarrow z^* = 4 - 3i$

ii) $z = 2 - 5i \Longrightarrow z^* = 2 + 5i$

6. Properties of Conjugate Complex Numbers :

Given $z = (a + ib)$ and $w = (c + id)$

then $z^* = a - ib$ and $w^* = c - id$

i) $(z^*)^* = z$

ii) $z + z^* = 2\,\text{Re}(z)$

iii) $z - z^* = 2i\,\text{Im}(z)$

iv) $z = z^* \Longleftrightarrow z$ is pure real

v) $z + z^* = 0 \Longleftrightarrow$ z is pure Ima.

vi) $z.z^* = (\text{Re } z)^2 + (\text{Im} z)^2 = a^2 + b^2$

or $zz^* = |z|^2$

vii) $(z + w)^* = z^* + w^*$

viii) $(z - w)^* = z^* - w^*$

ix) $(zw)^* = z^*w^*$

x) $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$; $w \neq 0$

xi) A quadratic equation with **real** coefficients :

$ax^2 + bx + c = 0$ such that $b^2 - 4ac < 0$

has conjugate complex roots.

Example : $z^2 + 4z + 13 = 0$ has conjugate complex roots

i.e $(-2 + 3i)$ and $(-2 - 3i)$

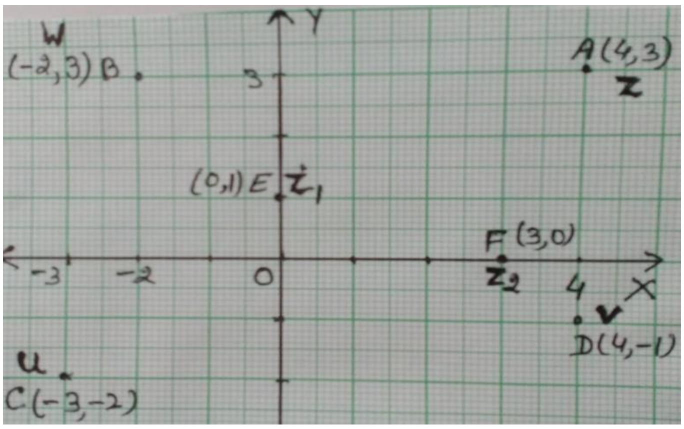
6. Geometric Representation of a Complex Numbers

To each complex numbers $z = (x + iy)$ there corresponds a unique ordered pair (a, b) or a point $A(a, b)$ on Argand diagram

Example : Represent the following complex numbers on an Argand Diagram :

i) $z = (4 + 3i)$ ii) $w = -2 + 3i$ iii) $u = (-3 - 2i)$ iv) $v = (4 - i)$

v) $z_1 = i$ vi) $z_2 = +3$



Solution: i) $z = (4 + 3i) \rightarrow A(4,3)$

ii) $w = (-2 + 3i) \rightarrow B(-2, 3)$

iii) $u = (-3 - 2i) \rightarrow C(-3, -2)$

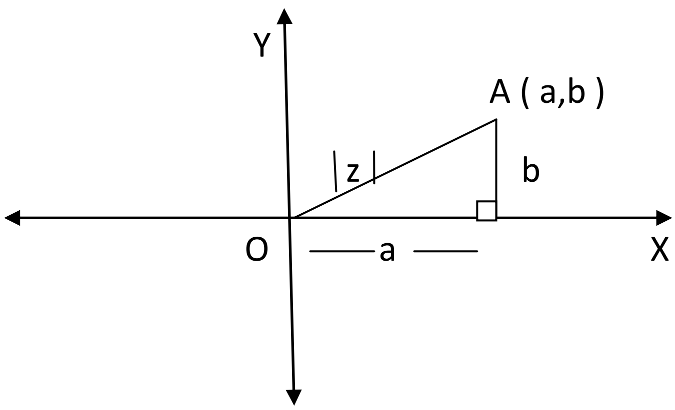
iv) $v = (4 - i) \rightarrow D(4, -1)$

v) $z_1 = i \rightarrow E(0,1)$

vi) $z_2 = 3 \rightarrow F(3, 0)$

7. Modulus of a Complex Numbers :

$Z = (a + ib) \Rightarrow A(a, b)$



Modulus of z = $|z| = OA$

$= \sqrt{(a^2 + b^2)}$; $|z| \geq 0$

Example : $z = (4 + 3i) \Rightarrow |z| = \sqrt{(4^2 + 3^2)}$

$= 5$

NOTE : $z = a + ib$, $z^* = (a - ib)$

$zz^* = (a + ib)(a - ib)$

$= a^2 + b^2$

$= (\sqrt{(a^2 + b^2)})^2 = |z|^2$

$zz^* = |z|^2$

8. Properties of Modulus of a Complex Numbers

$z, z_1, z_2 \in C$

i) $|z| = 0 \Leftrightarrow z = 0$ or (Re z = 0 and Ima z = 0)

ii) $|z| = |z^*| = |-z|$

iii) $-|z| \leq \text{Re } z \leq |z|$ and $-|z| \leq \text{Ima } z \leq |z|$

iv) $zz^* = |z|^2$

v) $|z_1 z_2| = |z_1| \, |z_2|$

vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$

vii) $\frac{1}{z} = \frac{z^*}{|z|^2}$

viii) $|z^2| = |z|^2$

9. To find square roots of a complex number (a + i b)

Let (x + i y) is square root of (a + i b)

$(x + i y)^2 = (a + b i)$ ________________ (i)

Or $(x^2 - y^2) + 2xyi = (a + b i)$

Or $x^2 - y^2 = a$ ____________ (ii)

$2xy = b$ ____________ (iii)

Taking modulus on both sides of (i)

$|(x + i y)^2| = |a + i b|$

$|x + i y|^2 = \sqrt{(a^2 + b^2)}$

Or $x^2 + y^2 = \sqrt{(a^2 + b^2)}$ __________ (iv)

Adding equation (ii) and (iv) we get,

$2x^2 = (a + \sqrt{(a^2 + b^2)})$

$x^2 = \frac{1}{2}(a + \sqrt{(a^2 + b^2)}) = p$ (let)

$x = \pm \sqrt{p}$ __________ (v)

Now subtract equation (ii) from (iv)

$2y^2 = \sqrt{(a^2 + b^2)} - a$

$y^2 = \frac{1}{2}(\sqrt{(a^2 + b^2)} - a) = q$ (let)

$y = \pm \sqrt{q}$

<u>NOTE :</u> Case I : Now from equation (iii) if $2xy = b > 0$

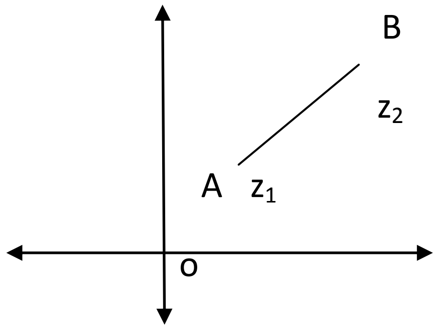
Then req. Square roots $= \pm (\sqrt{p} + i \sqrt{q})$

Case II : If $2xy = b < 0$

Then req. Square roots $= \pm (\sqrt{p} - i \sqrt{q})$

10. Important application of modulus of complex numbers :

If Complex numbers z_1 and z_1 are represented by points A and B respectively.



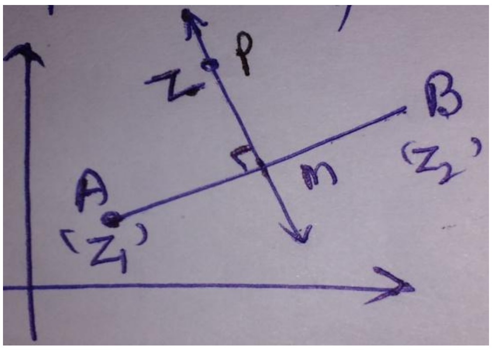
i) $\overrightarrow{AB} = z_2 - z_1$

and $|z_2 - z_1| = AB$

and $|z_1| = OA$

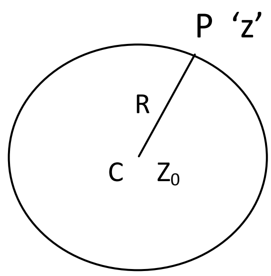
ii) Given $|z - z_1| = |z - z_2|$ represents the locus of z, which is the set of points **P** equidistant from two given points A 'z_1' and B 'z_2'

Hence , Locus of z is the perpendicular bisector of AB



iii) Equation of circle

a) $|z - z_0| = R$



Locus of z is a circle :

Centre at C ‘ z_0 ‘ and radius CP = R

b) **Show that** : Complex equation :

$z\ z^* + a z^* + a^* z + b = 0$ ....(i) ; $a \varepsilon C$, $b \varepsilon R$
represent a circle with centre at ‘-a ‘ and radius $R = \sqrt{(|a|^2 - b)}$

Solution : from equation (i)

$$z\ z^* + a z^* + a^* z = - b$$

Add aa^* on both the sides

$$z z^* + a z^* + a^* z + a a^* = - b + a a^*$$

$$\Longrightarrow \quad z^* (z + a) + a^* (z + a) = a a^* – b$$

$$\Longrightarrow \quad (z + a)\ (z^* + a^*) = |a|^2 - b$$

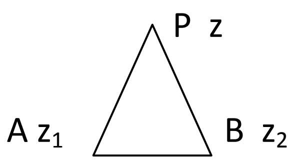
$$(z + a)\ (z + a)^* = |a|^2 - b$$

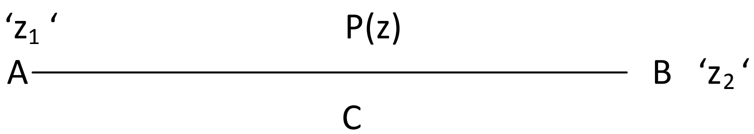
$$|z + a|^2 = |a|^2 - b$$

$$|z + a| = \sqrt{(|a|^2 - b)} \text{ ----------------------------(ii)}$$

Comparing it with { $|z - a| = R$ } equation (ii) represent a circle with centre at ‘ – a ’ and $R = \sqrt{(|a|^2 - b)}$

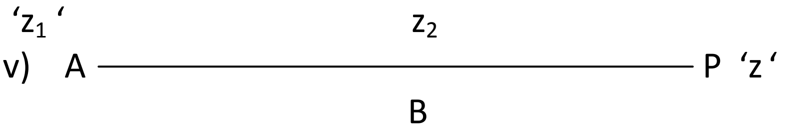
c) $| z – z_1 | = k \ | z – z_1 |$; k R^+ , $k \neq 1$ represent a circle , $\frac{PA}{PB} = k$



iv) 

$|z - z_1| + |z - z_2| = |z_1 - z_2|$ (or PA + PB = AB)

P lies on segment AB

v) 

$|z - z_1| - |z - z_2| = |z_1 - z_2|$ or PA - PB = AB

P lies on segment $\overrightarrow{BP}$

11) Argument of a Complex Numbers :

z = (a + b i) complex number is represented by point A

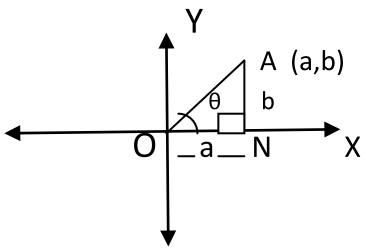
arg z = $\angle AOX = \theta$

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

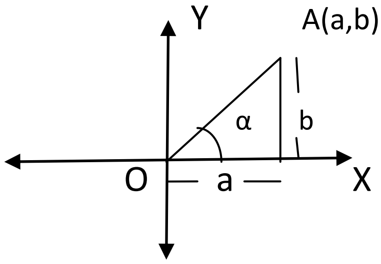
$-\pi < \theta \leq \pi$

Here θ is principal arguments of z.



CASE I a> 0 , b > 0

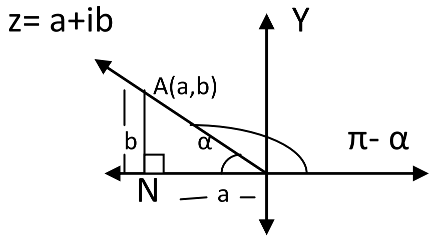
$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \alpha$



CASE II a <0 and b>0

$\tan^{-1}\left|\frac{b}{a}\right| = \alpha$

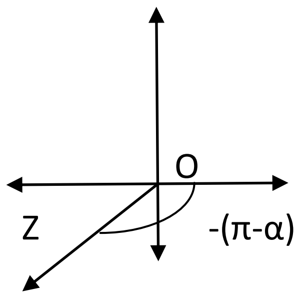
$\theta = (\pi - \alpha)$



CASE III a<0 and b< 0

$\tan^{-1} \left| \frac{b}{a} \right| = \alpha$

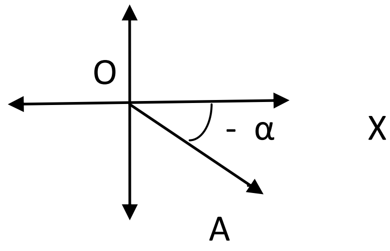
$\theta = -(\pi - \alpha)$



CASE IV a>0 and b< 0

$\tan^{-1} \left| \frac{b}{a} \right| = \alpha$

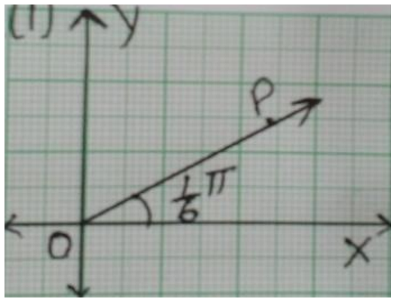
$\theta = -\alpha$



12. Application of argument of Complex numbers

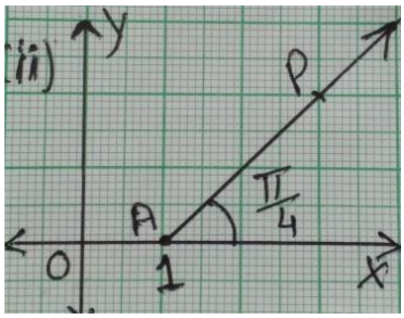
i) $\arg z = \frac{\pi}{6}$

Locus of z is half line $\overrightarrow{OP}$



ii) $\arg (z - 1) = \frac{\pi}{4}$

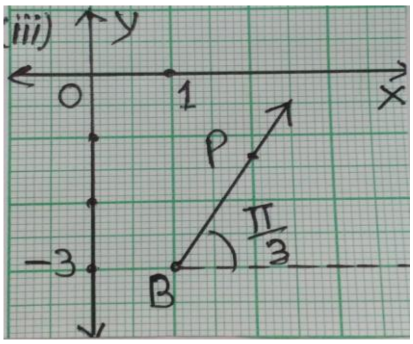
Locus of z is half line $\overrightarrow{AP}$



iii) $\arg (z - 1 + 3i) = \frac{\pi}{3}$

or $\arg [z - (1 - 3i)] = \frac{\pi}{3}$

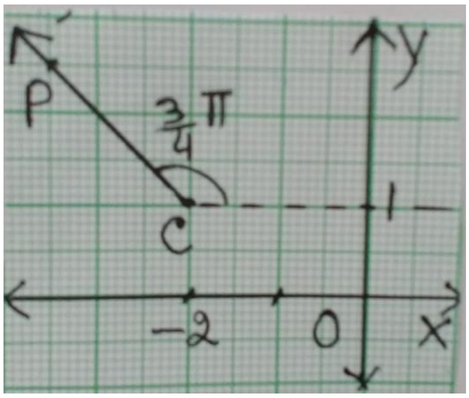
Locus of z is half line $\overrightarrow{BP}$



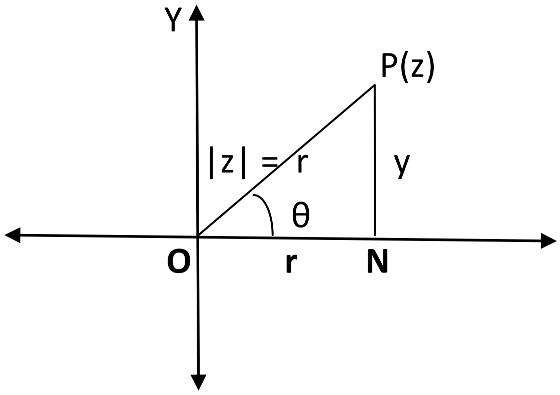
iv) $\arg(z + 2 - i) = \frac{3\pi}{4}$

or $(z - (-2 + i)) = \frac{3\pi}{4}$

Locus of z is half line $\overrightarrow{CP}$



13. <u>Polar form of a Complex Numbers</u>



Given a Complex number $z = x + iy$

$|z| = \sqrt{x^2 + y^2} = r$ (let)

And $\arg z = \tan^{-1}\frac{y}{x} = \theta$

Then Polar form of z

$$z = r(\cos\theta + i\sin\theta)$$

14. <u>Multiplication of Complex Numbers in Polar form:</u>

Given complex numbers

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then, $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Note : i) $|z_1 z_2| = |z_1|\,|z_2| = = r_1 r_2$

ii) $\arg(z_1 z_2) = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2 + K(2\pi)$

$$K = \begin{cases} 0 & \text{if } -\pi < (\theta_1 + \theta_2) \le \pi \\ -1 & \text{if } (\theta_1 + \theta_2) > \pi \\ +1 & \text{if } (\theta_1 + \theta_2) < -\pi \end{cases}$$

15. **<u>Division of Complex Numbers In Polar Form:</u>**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \quad ; \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$\theta = (\arg z_1 - \arg z_2) + K\,(2\pi)$$

$$\begin{cases} K = 0 & \text{if } -\pi < \theta_1 - \theta_2 \le \pi \\ -1 & \text{if } (\theta_1 - \theta_2) > \pi \\ +1 & \text{if } (\theta_1 - \theta_2) < -\pi \end{cases}$$

16. Square root of a Complex Number in Polar Form

Given Complex number $z = r\,(\cos\theta + i\sin\theta)$

Let the square root of z is $w = p\,(\cos\alpha + i\sin\alpha)$ ____________ (i)

$[\,p(\cos\alpha + i\sin\alpha)\,]^2 = r\,(\cos\theta + i\sin\theta)$

Or $p^2\,(\cos 2\alpha + i\sin 2\alpha) = r\,(\cos\theta + i\sin\theta)$

$p^2 = r \Rightarrow p = \sqrt{r}$

and $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$

i) arg w

$$\alpha = \begin{cases} \frac{\theta}{2} \text{ and } \frac{\theta}{2} - \pi & \text{if } \theta \ge 0 \\ \quad\text{or} & \\ \frac{\theta}{2} \text{ and } \frac{\theta}{2} + \pi & \text{if } \theta < 0 \end{cases}$$

17. Exponential form a complex Number :

Given a Complex number in Polar form $z = r\,(\cos\theta + i\sin\theta)$

Then exponential form $z = re^{i\theta}$