Pure Maths 3

Complex Numbers
Revision

Sureesh Gael  
(former Director)  
Alliance World School  
Noida, Delhi - NCR  
INDIA.
Example 1 (a) The complex numbers \( v \) and \( w \) satisfy the equations:
\[
v + iw = 5 \quad \text{and} \quad (1+2i)v - w = 3i
\]
Solve the equations for \( v \) and \( w \), giving your answers in the form \( x+iy \), where \( x \) and \( y \) are real.

(b) (i) On an Argand diagram, sketch the locus of points representing complex numbers \( z \) satisfying:
\[
|z - 2 - 3i| = 1
\]
(ii) Calculate the least value of \( \arg z \) for the points on the locus.

Solution (a) Solve for \( v \) and \( w \) (use \( i^2 = -1 \))

\[
v = -2i \quad \text{and} \quad w = 5 + 7i
\]

Multiply \( N'v \) and \( Z'v \) by the conjugate of \( v \).

We get:
\[
v = -1 - i \quad \checkmark
\]
and
\[
w = 1 - 6i \quad \checkmark
\]

(b) (i) \( |z - 2 - 3i| = 1 \)
or \( |z - (2+3i)| = 1 \)
represents a circle with center at \( (2+3i) \) and \( r = 1 \).

Draw \( OP \) tangent to the circle.

The angle \( \theta = \angle POX = \theta \) is the least.

\[
\theta = \arg(2+3i) - \alpha
\]
\[
= \tan^{-1} \frac{3}{2} - \tan^{-1} \frac{1}{\sqrt{3}}
\]
\[
= 56.3^\circ - 16.4^\circ
\]
\[
= 40.9^\circ \quad \checkmark
\]
Example (ii): The complex number \( u \) is defined by
\[
\frac{3i}{a+2i}
\]
where \( a \) is real.

(a) Express \( u \) in the Cartesian form \((x+i)\), where \( x \) and \( y \) are in terms of \( a \). \(-3\)

(b)(i) Find the exact value of \( a \) for which \( \arg u = \frac{\pi}{3} \). \(-3\)

(b)(ii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers \( z \) satisfying the inequalities \(|z-2i| < |z-1-i| \) and \(|z-2i| < 2 \). \(-4\)

(iii) Calculate the least value of \( \arg z \) for the points in this region. \(-2\)

Solution

(i) Multiply \( N^2 \) and \( Z^3 \) by \((a-2i)\) and use \( i^2 = -1 \)

\[
\frac{u}{(a+4)} = \frac{6}{a+4} - \frac{3ai}{a+4} \Rightarrow \arg u = \tan^{-1}\left( \frac{-3a}{6} \right) = \frac{\pi}{3} \text{ (given)} \]

\[
\Rightarrow -\frac{3a}{6} = \frac{\sqrt{3}}{3} \]

\[
\Rightarrow a = -2\sqrt{3}
\]

(b)(i) \(|z-2i| = |z-(1+i)|\) represent the points joining \( A(2i), B(1+i) \).

and \(|z-(2+i)| = 2 \) represents a circle centred \((2+i)\) and \( r = 2 \).

(ii) Least value of \( \arg z \) in the shaded region at \( P(2+3i) \)

\[
\arg (2+3i) = \tan^{-1}\left( \frac{3}{2} \right) = 56.3^\circ
\]
Example 3 (a) Solve the equation \((1+2i)w + iw^2 = 3+5i\). Give your answer in the form \(x+iy\), where \(x\) and \(y\) are real. \(-[4]\)

(b) i. On a sketch of Argand diagram, shade the region whose points represent complex numbers \(z\) satisfying the inequalities:

\[|z-2-2i|\leq 1 \text{ and } \arg(z-4i)\leq -\frac{\pi}{4}\] \(-[4]\)

ii. Find the least value of \(\text{Im}\,\bar{z}\) for the points in this region, giving your answer in an exact form. \(\frac{5-20\sqrt{2}}{32}\) \((-2\sqrt{2})\)

Solution (a) \((1+2i)(x+iy)+i(x-iy)=3+5i\)  

Equate real and imaginary parts; \([u+i^2 = -1]\)

\[x-y = 3 \quad \text{and} \quad 3x+5 = 5\]

Solve for \(x\) and \(y\) and get \(w = (2-i)\)

(b) i. \(|z-(2+2i)|\leq 1\); represents a circle centre \((2+2i)\), \(r=1\)

\((z-4i)\geq -\frac{\pi}{4}\) represents a half-line from \(4i\).  

Shade the correct region.

(ii) \(A\) is the point with the least \(\text{Im}\,\bar{z}\). In the shaded area,

Draw \(AM\perp X\)-axis

\[CN\perp X\text{-axis}\]

\[AD \perp CN\]

In \(\triangle CAD\),

\[\frac{CD}{CA} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{1}{\sqrt{2}} \quad (CA=x=1)\]

\[CD = CA \times \frac{\sqrt{2}}{2} = \frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\]

Now, \(AM = DN = CN - CD\)

\[= 2 - \frac{1}{\sqrt{2}}\]

\[\therefore \text{Im}\,\bar{z} = 2 - \frac{1}{\sqrt{2}} = \left(2 - \frac{1}{2\sqrt{2}}\right)\]
Example 4 (a) Complex numbers \( u \) and \( w \) are such that:
\[
u - w = 2i \quad \text{and} \quad uw = 6
\]
Find \( u \) and \( w \), giving your answers in the form \( x + iy \), where \( x \) and \( y \) are real and exact. \(-[5]\)

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers \( z \) satisfying the inequalities:
\[
|z - 2 - 2i| \leq 2, \quad 0 \leq \arg z \leq \frac{1}{4} \pi \quad \text{and} \quad \Re z \leq 3.
\]
\([5-20/33/09]\)

Solution (a) \( u - w = 2i \) and \( uw = 6 \)

Eliminating \( w \), \( \Rightarrow u^2 - 2i \cdot u - 6 = 0 \)

\[
\Rightarrow u = \sqrt{5} + i, \quad w = \sqrt{5} - i
\]

or \( u = -\sqrt{5} + i, \quad w = -\sqrt{5} - i \)

(b) \( |z - (2+2i)| = 2 \), represent a circle, centre \((2,2)\), rad = 2

\( 0 \leq \arg z \leq \frac{1}{4} \pi \), show half line from origin at \( 45^\circ \) to the positive \( x \)-axis

\( \Re z \leq 3 \) represent a line parallel to \( y \)-axis, through \( x = 3 \)
Example 5 (a) Showing all working and without using a calculator, solve the equation: 
\[(1+i)z^2 - (4+3i)z + 5 + i = 0\]
Give your answers in the form \(x+iy\), where \(x\) and \(y\) are real.

(b) The complex number \(u\) is given by:
\[u = -1 - i\]

On a sketch of an Argand diagram show the point representing \(u\). Shade the region whose points represent complex numbers satisfying the inequalities \(|z| < |z-2i|\) and \(\frac{\pi}{2} < \arg(z-u) < \frac{3\pi}{4}\)

Examples (a), using quad. formula,
get the final answers, \((-1 - i), (\frac{5}{2} + \frac{1}{2}i)\)

(b) \(u = -1 - i\) represents point A, \((-1, -1)\)

| \(z - 0| = |z - 2i|\)

represent line \(l\) - perp. bisector.

\(z = i\) \(\text{def. join} z = 0 \& z = 2i\)

\(\arg(z - u) = \frac{\pi}{4}\)

\(\frac{\pi}{2}\)

\(-\frac{\pi}{2}\)

\(0\)

\(z = 0\)

\(z = 2i\)

\(z = i\)

\(z = -1\)

\(z = -i\)

\(u\)

\(A(-1, -1)\)
Example 6: The complex number \( \sqrt{3} + i \) is denoted by \( z \).

(i) Express \( z \) in the form \( r e^{i\theta} \), where \( r > 0 \) and \(-\pi < \theta < \pi\), giving the exact values of \( r \) and \( \theta \). Hence or otherwise state the exact values of the modulus and argument of \( z^4 \). --- [5]

(ii) Verify that \( z \) is a root of the equation \( z^3 - 8z + 8\sqrt{3} = 0 \) and state the other complex root of this equation. --- [3]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex number \( z \) satisfying the inequalities \( |z - 1| \leq 2 \) and \( \text{Im} z \geq 2 \), where \( \text{Im} z \) denotes the imaginary part of \( z \).

\[
\begin{align*}
(ii) \quad z &= (\sqrt{3} + i) \\
&= 2e^{i\pi/6} \\
&= 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \\
&= 2 \left( \frac{\sqrt{3}}{2} + i \frac{1}{2} \right) \\
&= \sqrt{3} + i
\end{align*}
\]

Consider \( z^3 - 8z + 8\sqrt{3} = 0 \)

Put \( z = u \), \( u^3 - 8u + 8\sqrt{3} = 0 \)

\( u = \sqrt{3} \)

\( u^3 = 8(\sqrt{3} + i) + 8\sqrt{3} = 0 \)

\( u \) is a root if \( z^3 - 8z + 8\sqrt{3} = 0 \)

Other root will be \( u^4 = \sqrt{3} - i \)

\[
\begin{align*}
(iii) \quad |z - 1| &\leq 2 \\
|2 - (\sqrt{3} + i)| &\leq 2 \\
\text{Circle Centre} (\sqrt{3}, 1), \text{Radii} = 2 \\
\text{Im} z = 2, \text{line parallel to x-axis} \\
\text{Im} z &\geq 2
\end{align*}
\]
Example 7: It is given that the complex number $-1+\sqrt{3}i$ is a root of the equation $kx^3+5x^2+10x+4=0$ where $k$ is a real constant.

(i) Write down another root of the equation.

(ii) Find the value of $k$ and the third root of the equation.

Solution:

(i) Given a root $-1+\sqrt{3}i$, then the other root will be its conjugate, $-1-\sqrt{3}i$.

(ii) Now let $x = -1+\sqrt{3}i \Rightarrow x^2 = (-1+\sqrt{3}i)^2 = 1 - 2\sqrt{3}i$  
\[\quad \quad = -2 + 2\sqrt{3}i \checkmark\]

and $x^3 = (-1+\sqrt{3}i)^3 = (-1)^3 + (\sqrt{3}i)^3 + 3(-1)(\sqrt{3}i)^2 + 3(-1)^2 \sqrt{3}i$  
\[\quad \quad = -1 - 3\sqrt{3}i + 9 + 3\sqrt{3}i = 8 \checkmark\]

Given $kx^3+5x^2+10x+4=0$  \[\quad \quad \Rightarrow k \times 8 + 5(-2-2\sqrt{3}i)+10(-1+\sqrt{3}i)+4=0\]
\[8k-16=0 \Rightarrow k=2 \checkmark\]

Now $x = (-1+\sqrt{3}i)$ and $x = (-1-\sqrt{3}i)$

$\Rightarrow (x+1-\sqrt{3}i)(x+1+\sqrt{3}i)$ is a factor of eqn.  \[\checkmark\]
\[\Rightarrow (x^2+2x+4)$ is a factor of \[\checkmark\]
\[\Rightarrow 2x^3+5x^2+10x+4=0 $ \quad (h=2)$ in eq.  \[\checkmark\]
\[\Rightarrow (x^2+2x+4)(2x+1)=0\]
\[\Rightarrow (2x+1)=0\]
\[\Rightarrow x=-\frac{1}{2}$ is the third root.
Example 8: The complex number \( U \) is defined by \( U = \frac{4i}{1-\sqrt{3}i} \).

(i) Express \( U \) in the form \( x + iy \), where \( x \) and \( y \) are real and exact.

(ii) Find the exact modulus and argument of \( U \).

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers \( Z \) satisfying the inequalities \( |Z| < 2 \) and \( |Z-U| < |Z| \). 

Solution

(i) \( U = \frac{4i}{1-\sqrt{3}i} \) 

\[ \frac{4i(1+\sqrt{3}i)}{1+3} = \frac{4i(1+\sqrt{3}i)}{4} \]

\[ U = \frac{4}{4}(-\sqrt{3} + i) = -\sqrt{3} + i \sqrt{3} \]

(ii) \( |U| = |-\sqrt{3} + i\sqrt{3}| = \sqrt{3} + i \approx 2 \)

\[ \arg U = \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \]

(iii) \[ (-\sqrt{3}+i) \]

Note: The points on the boundary of the shaded region are not included.

\( |Z| < 2 \) represents the interior of a circle with centre at origin and rad. 2.

\( |Z-U| < |Z| \) To shade the 0-side of the perpendicular bisector of the segment joining the origin and \( U \).
Example 9(a) The complex number $u$ is given by $u = -3 - 2\sqrt{10}i$. Find the square root of $u$. Give your answers in the form $(a + ib)$, where the numbers $a$ and $b$ are real and exact. 

(b) On a sketch of an Argand diagram shade the region whose points represent complex numbers $z$ satisfying the inequalities $|z - 3 - i| \leq 3$, $\arg z \geq \frac{\pi}{4}$ and $\text{Im} z \geq 2$, where $\text{Im} z$ denotes the imaginary part of the complex number $z$. 

Solution (a) $(a + ib)^2 = -3 - 2\sqrt{10}i$ 

or $(a^2 - b^2) + 2abi = -3 - 2\sqrt{10}i$ 

Equalise real and imaginary parts 

$(a^2 - b^2) = -3$ --- (1) 

$2ab = -2\sqrt{10}$ --- (2) 

Tabling modulus on both sides 

$a^2 + b^2 = \sqrt{3^2 + (2\sqrt{10})^2} = 7$ --- (3) 

From (2) $ab = -\sqrt{10}$ 

and $a = \pm \sqrt{5}$ 

and $b = \pm \sqrt{5}$ 

required square roots are $\pm (\sqrt{5} - \sqrt{5}i)$ 

(b) $|z - 3 - i| \leq 3$ represents the interior of circle centre at $(3 + i)$ and $r = 3$. 

$\arg z \geq \frac{\pi}{4}$ represents a half line and its upper arc of 

a half line through origin. 

$\text{Im} z \geq 2$ is the right side of the line $y = 2$. 

$\text{Im} z = 2$ from Argand diagram.
Example 10 (a) Find the complex number \( z \) satisfying the equation
\[
\frac{z + iz}{z^*} - 2 = 0 \quad \text{where } z^* \text{ denotes the complex conjugate of } z.
\]
Give your answer in the form \( x + iy \), where \( x \) and \( y \) are real.

(b) (i) On a single Argand diagram sketch the loci given by the equations \( |z - 2i| = 2 \) and \( \text{Im} z = 3 \), where \( \text{Im} z \) denotes the imaginary part of \( z \).

(ii) In the first quadrant the two loci intersect at the point \( P \). Find the exact argument of the complex number represented by \( P \).

Solution (a) \( z + iz - 2 = 0 \)
\[
\Rightarrow (x + iy) + i(x + iy) - 2 = 0
\]
\[
\frac{x + iy}{x - iy} = 2
\]
\[
(x + iy)(x - iy) + i(x + iy) - 2(x - iy) = 0
\]
\[
x^2 + y^2 + i2xy - y - 2x + iy = 0
\]
\[
\{ x^2 + y^2 - 2x - y = 0 \quad \text{(0)} \}
\]
\[
x + 2y = 0 \quad \text{(2)}
\]

Solve (0) and (2)
\[
(-2y)^2 + y^2 - 2(-2y) - y = 0
\]
\[
y^2 + 3y = 0
\]
\[
y(3y + 3) = 0
\]
\[
y = -\frac{3}{5} \quad \text{or} \quad y = 0
\]
\[
x = \frac{6}{5}
\]
\[
\Rightarrow \text{Required } z = \left( \frac{6}{5} - \frac{3}{5}i \right)
\]

(b(i)) \( |z - 2i| = 2 \) represents a circle with centre at \( 2i \) and radius 2.

\[
\begin{align*}
\text{In } \triangle AOB, & \quad \rho B = \sqrt{3}\text{ }\text{and } \frac{\rho Q}{\rho B} = \frac{\sqrt{3}}{2} \\
\rho Q = \rho B = \sqrt{3} & \quad \rho Q = 0 \beta = 3
\end{align*}
\]
\[
\arg P = \frac{\rho Q}{\rho B} = \frac{\sqrt{3}}{2}
\]
\[
= \frac{\sqrt{3}}{\sqrt{3}} = \frac{\pi}{3}
\]
\[
\Rightarrow \arg P = \frac{\pi}{3} \left( 60^\circ \right)
\]
Example 11: The complex with modulus 1 and argument $\frac{1}{3} \pi$ is
denoted by $w$.

(i) Express $w$ in the form $x + iy$, where $x$ and $y$ are real and exact.
The complex number $1 + 2i$ is denoted by $u$. The
complex number $v$ is such that $|v| = 2|u|$ and
\[ \text{arg } v = \text{arg } u + \frac{\pi}{3} \]

(ii) Sketch an Argand diagram showing the points $u$ and $v$.

(iii) Explain why $v$ can be expressed as $2u$. Hence find $v$,
giving your answer in the form $a + ib$, where $a$ and $b$
are real and exact.

\[ |u| = 1, \quad \text{arg } w = \frac{1}{3} \pi \]

\[ w = \mathbf{z}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) \]

\[ w = \frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \Box \]

\[ u = 1 + 2i \]

\[ |u| = \sqrt{1^2 + 2^2} = \sqrt{5} \]

\[ \text{and } \text{arg } u = \tan^{-1} 2 = \tag{2} \]

Now $v$: \[ |v| = 2|u| = 2\sqrt{5} \quad \tag{3} \]

\[ \text{and } \text{arg } v = \text{arg } u + \frac{\pi}{3} \quad \tag{4} \]

\[ |2uw| = 2|u||w| = 2\sqrt{5} \times 1 = 2\sqrt{5} \quad \tag{5} \]

From (3) & (5), \[ |2uw| = |v| \quad \tag{6} \]

Now \[ \text{arg } (2uw) = \text{arg } (u) + \text{arg } (2w) \]
\[ = \text{arg } u + \text{arg } (1 + \sqrt{3} i) \]
\[ = \text{arg } u + \tan^{-1} \sqrt{3} \quad \tag{7} \]

From (4) and (7), \[ \text{arg } v = \text{arg } (2uw) \]

\[ \vdots \]

\[ v = 2uw = 2(1 + 2i)(\frac{1}{2} + \frac{\sqrt{3}}{2} i) \]
\[ = (1 - 2\sqrt{3}) + (2 + 3\sqrt{3}) i \]