A- LEVEL – MATHEMATICS P_3 Differential Equations

EXERCISE 1 (Answers on page 9-10)
(With References)

Q1. The variables x and θ satisfy the differential equation :

$$\frac{dx}{d\theta} = (x + 2)\sin^2 2\theta$$

and it is given that x = 0 when θ = 0. Solve the differential equation and calculate the value of x when $\theta = \frac{\pi}{4}$ giving your answer correct to 3 significant figures. **9**

[2017/ SP -3/Q8]
[W-15 /31/32/Q8]

Q2. The variables x and y satisfy the differential equation :

$\frac{dy}{dx} = x e^{x+y}$ and it is given that y = 0 when x=0

i) Solve the differential equation and obtain an expression for y in terms of x. **7**

ii) Explain briefly why x can only take values less than 1. **1**

[M-16 / 32 /Q7]

Q3. The variables x and y satisfy the differential equation:

$x\frac{dy}{dx} = y(1 - 2x^2)$ and it is given that y = 2 when x = 1.

Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. **6**

[S-16 / 31/Q4]

Q4. The variables x and θ satisfy the differential equation:

$(3 + \cos 2\theta)\frac{dx}{d\theta} = x.\sin 2\theta$ and it is given that x = 3 when $\theta = \frac{\pi}{4}$

i) Solve the differential equation and obtain an expression for x in terms of θ. **7**

ii) State the least value taken by x. **1**

[S-16/32/Q6]

Q5. The variables x and y satisfy the differential equation:

$$\frac{dy}{dx} = e^{-2y} \cdot \tan^2 x \quad , \text{for } 0 \le x < \frac{1}{2}\pi$$ and it is given that y = 0 when x = 0.

Solve the differential equation and calculate the value of y when $x = \frac{1}{4}\pi$ **8**

[S-16/33/Q5]

Q6. A large field of area 4 km^2 is becoming infected with a soil disease. At time t years the area infected is x km^2 and the rate of growth of the infected area is given by the differential equation:

$\frac{dx}{dt} = kx(4 - x)$, where k is a positive constant. It is given that when t = 0 , x = 0.4 and that when t = 2 , x = 2.

i) Solve the differential equation and show that $k = \frac{1}{4} \ln 3$. **9**

ii) Find the value of t when 90% of the field is infected. **2**

[W-16/31/Q10]

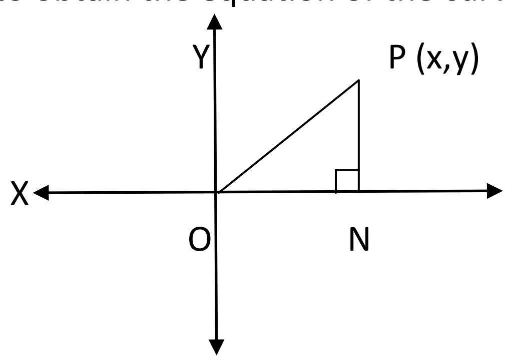
Q7. The diagram shows a variable point P with coordinate (x , y) and the point N which is the foot of the perpendicular from P to x – axis . P moves on a curve such that , For all x ≥ 0 , the gradient of the curve is equal in value to the area of the triangle 0PN, where O is origin.

i) State a differential equation satisfied by x and y. **1**

The point with coordinates (0, 2) lies on the curve.

ii) Solve the differential equation to obtain the equation of the curve , expressing y in terms of x. **5**

iii) Sketch the curve. **1**



[W-16/33/Q5]

Q8. Given that y = 1 when x = 0. Solve the differential equation

$\frac{dy}{dx} = 4x\,(3y^2 + 10y + 3)$

Obtaining an expression for y in terms of x. **9**

[S-15/31/Q7]

Q9. The number of organisms in a population at time t is given by x . Treating x as a continuous variable , the differential equation satisfied by x and t is:

$\frac{dx}{dt} = \frac{xe^{-t}}{k+e^{-t}}$ where k is a positive constant.

i) Given that x = 10 when t = 0 , solve the differential equation obtaining a relation between x, t and k. **6**

ii) Given also that x = 20 when t= 1, show that $k = 1 - \frac{2}{e}$ **2**

iii) Show that the number of organisms never reaches 48 , however large t becomes. **2**

[S-15/32/Q9]

Q10. The number of micro - organism in a population at time t is denoted by M. At any time the variation in M is assumed to satisfy the differential equation:

$\frac{dM}{dt} = K(\sqrt{M})\cos(0.02t)$, where K is constant and M is taken to be a continuous variable. It is given that when t = 0 , M =100.

i) Solve the differential equation obtaining a relation between M , K and t. **5**

ii) Given also that M = 196 when t = 50 , find the value of k. **2**

iii) Obtain an expression for M in terms of t and find the least possible number of micro- organisms. **2**

[S-15/33/Q7]

Q11. Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N , where N is treated as continuous variable.

i) It is given that the rate of increase of N with respect to t is proportional to (N – 150). Write down a differential equation relating N , t and a constant of proportionality. **1**

ii) Initially when t= 0 , the number of plants was 650. It is noted that at a time when there were 900 plants , the number of plants was increasing at a rate of 60 per year. Express N in terms of t. **7**

iii) The naturalist had a target of increasing the number of plants from 650 to 2000 within 15 years . Will this target be met? **2**

[W-15/33/Q10]

Q12. The variables x and y are related by the differential equation:

$$\frac{dy}{dx} = \frac{6\,y\,e^{3x}}{2+e^{3x}}$$

Given that y = 36 , when x = 0 , find an expression for y in terms of x. **6**

[S-14/31/Q4]

Q13. The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and (1 – 0.01N) when t= 0 , N = 20 and $\frac{dN}{dt} = 0.32$

i) Treating N and t as continuous variables, show that they satisfy the differential equation. $\frac{dN}{dt} = 0.02\,N\,(1 - 0.01N)$ **1**

ii) Solve the differential equation obtaining an expression for t in terms of N. **8**

iii) Find the time at which the population will be double its value at $t = 0$ **1**

[S-14/32/Q9]

Q14. The variables x and θ satisfy the differential equation :

$$2\cos^2\theta . \frac{dx}{d\theta} = \sqrt{(2x+1)}$$ **7**

And $x = 0$ when $\theta = \frac{1}{4}\pi$. Solve the differential equation and obtain an expression for x in terms of θ.

[S-14/33/Q5]

Q15. In a certain country government charges tax on each litre of patrol sold to motorist . The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation:

$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right)$, where R and x are taken to be continuous variables when $x = 0.5$, $R = 16.8$

i) Solve the differential equation and obtain an expression for R in terms of x. **6**

ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. **3**

[W-14/31/ 32/Q7]

Q16. The variable x and y are related by differential equation:

$$\frac{dy}{dx} = \frac{1}{5} x\, y^{\frac{1}{2}} . \sin\left(\frac{1}{3}x\right)$$

i) Find the general solution giving y in terms of x. **6**

ii) Given that $y = 100$ when $x = 0$, find the value of y when $x = 25$. **3**

[W-14 /33/Q8]

Q17. Liquid is flowing into a small tank which has a leak. Initially the tank is empty and t minutes later , the volume in the tank is V cm^3 .The liquid is flowing into the tank at a constant rate of 80 cm^3 per minute. Because of the leak , liquid is being lost from the tank at a rate which at any time is equal to KV cm^3 per minute where K is a positive constant.

i) Write down a differential equation describing this situation and solve it to show that:

$$V = \frac{1}{K}\left(80 - 80\, e^{-kt} \right)$$ **7**

ii) It is observed that V = 500 when t = 15, so that K satisfies the equation

$$K = \frac{4 - 4\, e^{-15k}}{25}$$ **3**

Use an iteration formula, based on this equation find the value of K correct to 2 significant figures. Use an initial value of K = 0.1 and show the result of each iteration to 4 significant figures.

iii) Determine how much liquid there is in the tank 20 minutes after the liquid started flowing and state what happens to the volume of liquid in the tank after long time. **2**

[S-13/31/Q10]

Q18. i) Express $\frac{1}{x^2\,(2\,x+1)}$ in the form $\frac{A}{x^2} + \frac{B}{x} + \frac{C}{(2x+1)}$ **4**

ii) The variables x and y satisfy the differential equation: **7**

$$y = x^2\,(2x + 1)\,\frac{dy}{dx}$$

and y = 1 when x= 1. Solve the differential equation and find the exact value of y when x= 2. Give your value of y in a form not involving logarithms.

[S-13/32/Q8]

Q19. The variable x and t satisfy the differential equation:

$$t.\frac{dx}{dt} = \frac{k - x^3}{2x^2}$$

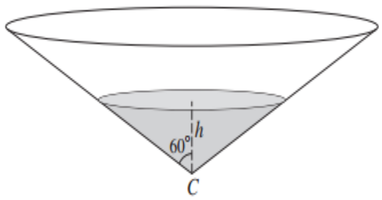
for $t > 0$, where k is a constant. When $t = 1$, $x= 1$ and when $t = 4$, $x = 2$

i) Solve the differential equation finding the value of k and obtaining an expression for x in terms of t. **9**

ii) State what happens to the value as t becomes large. **1**

[S-13/33/Q8]

Q20. A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi vertical angle is 60^0 as shown in the diagram. At $t = 0$, the tank is full and the depth of water is H . At this instant a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.



i) Show that h and t satisfy a differential equation of the form:

$$\frac{dh}{dt} = -A\ h^{\frac{-3}{2}}$$ where A is a positive constant . **4**

ii) Solve the differential equation given in part (i) and obtain an expression for t in terms of h and H. **6**

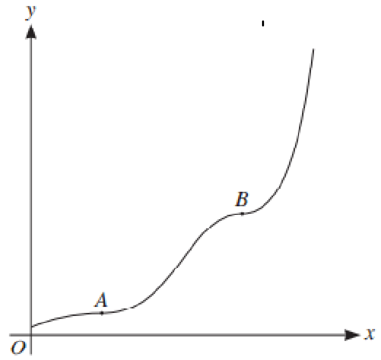
iii) Find the time at which the depth reaches $\frac{1}{2}$H. **1**

[W-13/31/32/Q10]

Q21. A particular solution of the differential equation:

$$3y^2 \frac{dy}{dx} = 4\ (\ y^3 + 1)\ \cos^2 x$$

is such that y = 2 when x= 0. The diagram shows a sketch of the graph of this solution of $0 \le x \le 2\pi$; the graph has stationary points at A and B. Find the y- coordinates of A and B , giving each coordinates to 1 decimal plane. **10**



[W-13/33/Q10]

Q22. The variables x and y are related by the differential equation:

$$\frac{dy}{dx} = \frac{6\,x\,e^{3x}}{y^2}$$ It is given that y = 2 when x=0.

Solve the differential equation and hence find the value of y when x = 0.5 giving yours answer correct to two decimal places. **8**

[S-12/31/Q7]

Q23. The variables x and y satisfy the differential equation:

$$\frac{dy}{dx} = e^{2x+y}$$ and y =0 when x=0. **6**

Solve the differential equation , obtain an expression for y in terms of x.

[S-12/32/Q5]

Q24. In a certain chemical process a substance A reacts with another substance B . The masses in grams of A and B present at time t seconds after the start of the process are x and y respectively. It is given that :

$$\frac{dy}{dt} = -0.6\,xy$$ and $x = 5e^{-3t}$, when t = 0 and y= 70

i) Form a differential equation in y and t. Solve this differential equation and obtain an expression for y in terms of t. **6**

ii) The percentage of initial mass of B remaining at time t is denoted by p. Find the exact value approached by p as t becomes large. **2**

[S-12/33/Q5]

Q25. The variable x and y are related by differential equation: **8**

$$x\frac{dy}{dx} = 1-y^2 \quad \text{when} \quad x = 2 \ , y = 0.$$

Solve the differential equation obtaining an expression for y in terms of x.

[W-12/31/32/Q6]

Q26. The variable x and y are related by differential equation: **6**

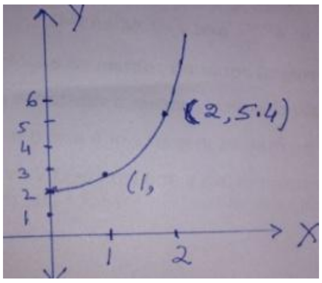
$$(x^2+4)\frac{dy}{dx} = 6xy \quad , \text{when } x = 0 \ , y = 32$$

Find the expression for y in terms of x.

[W-12/33/Q4]

A- LEVEL – MATHEMATICS **P_3** **EXCERCISE 1/ ANSWERS**

Differential Equation

Q1. i) $\ln\left(\frac{x+2}{2}\right) = \frac{\theta}{2} - \frac{1}{8}\sin 4\theta$ ii) $x = 0.962$	Q7. i) $\frac{dy}{dx} = \frac{1}{2}xy$ ii) $y = 2e^{\frac{1x^2}{4}}$ iii) 
Q2. i) $y = -\ln((1-x)e^x)$ ii) ln (1-x) is def when 1-x > 0 or x<1	Q8. $Y = \frac{3e^{16x^2}-1}{3-e^{16x^2}}$
Q3. $y = 2x \cdot e^{(1-x^2)}$	Q9. i) $x = \frac{10(k+1)}{(k+e^{-t})}$ ii) $K = \left(1 - \frac{2}{e}\right)$ iii) $x = \frac{10[2-\frac{2}{e}]}{(1-\frac{2}{e})}$ [$t \to \infty$ $e^{-t} \to 0$ $= 47.84 < 48$ as $t \to \infty$
Q4. i) $x = \sqrt{\left(\frac{27}{3+\cos 2\theta}\right)}$ ii) $x = \frac{3\sqrt{3}}{2}$ or (2.60)	Q10. i) $\sqrt{M} = 25K\sin(0.02t) + 10$ ii) k =0.190 iii) $M = [4.75\sin(0.02t)+10]^2$ and least value of M = 27.6
Q5. i) $y = \frac{1}{2}\ln[2(\tan x - x) + 1]$ ii) $y = 0.179$	Q11. i) $\frac{dN}{dt} = K(N-150)$ ii) $N = 500e^{0.08t} + 150$ iii) when t = 15, N = 1810 hence target of N= 2000 not met.
Q6. i) $\ln\left(\frac{x}{4-x}\right) = t.\ln 3 - \ln 9$ ii) $t = 4$	Q12. $y = 4(2 + e^{3x})^2$

Q13. i) $\frac{dN}{dt}$ = 0.02 N (1- 0.01 N) ii) t = 50 ln ($\frac{4N}{100-N}$) iii) t = 49	Q19. i) K = 9 and x = $[\ 9 - 8\ t^{\frac{-3}{2}}\ ]^{\frac{1}{3}}$ ii) $9^{\frac{1}{3}}$ when t is large.
Q14. x = $\frac{1}{8}$ $(1 + \tan\theta)^2$ - $\frac{1}{2}$	Q20. ii) t = 60 [1- $(\frac{h}{H})^{\frac{5}{2}}$] iii) t = 49.4
Q15. i) R = $x\,e^{(3.80 - 0.57x)}$ or = $44.7xe^{-0.57x}$ ii) R = 28.9	Q21. ii) stationary points at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ y = $[\ 9\ e^{(2x + \sin 2x)} - 1]^{\frac{1}{3}}$ At A when x= $\frac{\pi}{2}$, y = 5.9 At B when x= $\frac{3\pi}{2}$, y = 48.1
Q16. i) y = $[\ \frac{-3}{10} x \cos\frac{x}{3} + \frac{9}{10}\sin\frac{x}{3} + K\ ]^2$ ii) K=10 y = $[\frac{-3}{10} x \cos\frac{x}{3} + \frac{9}{10}\sin\frac{x}{3} + 10]^2$ and y = 203 when x = 25	Q22. y = $[\ 2\ e^{3x}(3x-1) + 10]^{\frac{1}{3}}$ And y = 2.44 when x= 0.5
Q17 i) V = $\frac{1}{K}(80 - 80\ e^{-kt})$ ii) K = 0.14 iii) V = 536.67	Q23. y = ln [$\frac{2}{3-e^{2x}}$]
	Q24. i) y = $70\ e^{(e^{-3t} - 1)}$ ii) y = $\frac{70}{e}$
	Q25. y = $\frac{x^2-4}{x^2+4}$
Q18. i) A=1 , B = - 2 , C =4 ii) ln y = 1 - $\frac{1}{x}$ + 2 ln ($\frac{2x+1}{3x}$) and y = $\frac{25}{36}e^{\frac{1}{2}}$ when x = 2	Q26. y = $\frac{1}{2}$ $(x^2 + 4)^3$