A – Level - Mathematics **P_3**

ITERATION (With references and answers)

[Numerical Solution of Equation]

Q1. The equation $x^3 - x^2 - 6 = 0$ has one real root , denoted by α .

i) Find by calculation the pair of consecutive integers between which α lies. [2]

ii) Show that , if a sequence of values given by iterative formula:

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n} \right)}$$ converges , then it converges to α. [2]

iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(2017 / SP -3 /Q4)
(W-15/31/32/Q4)

Q2. The equation $x^5 - 3x^3 + x^2 - 4 = 0$ has one positive root.

i) Verify by calculation that this root lies between 1 and 2. [2]

ii) Show that the equation can be rearranged in the form:

$$x = \sqrt[3]{\left(3x + \frac{4}{x^2} - 1 \right)}$$ [1]

iii) Use an iteration formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(M-16 / 32 / Q3)

Q3. i) By checking a suitable pair of graphs , show that the equation:

$$5e^{-x} = \sqrt{x}$$ has one root. [2]

ii) Show that, if a sequence of values given by the iterative formula,

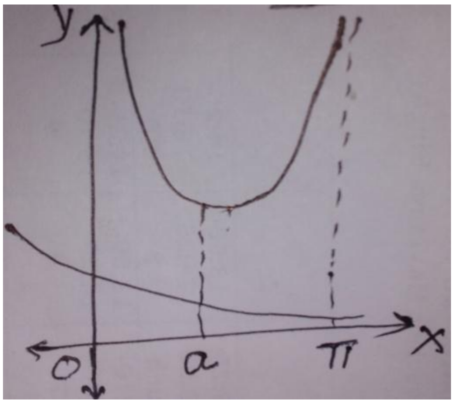
$$x_{n+1} = \frac{1}{2} \ln \left(\frac{25}{x_n} \right)$$ [2]

converges , then it converges to the root of the equation in part (i)

iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(S-16 / 31 / Q6)

Q4. The diagram shows the curve $y = \operatorname{cosec} x$, for $0 < x < \pi$ and the part of the curve $y = e^{-x}$ when $x = a$, the tangents to the curves are parallel.



i) By differentiating $\frac{1}{\sin x}$, show that :

if $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$

[3]

ii) By equating the gradient of the curves at $x = a$,

show that :

$$a = \tan^{-1}\left(\frac{e^a}{\sin a}\right)$$ [2]

iii) Verify by calculation that a lies between 1 and 1.5 [2]

iv) Use an iterative formula based on an equation in part (ii) to determine **a ,** correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(S-16 / 32 / Q8)

Q5. The curve with equation $y = x^2 \cos\frac{x}{2}$ has stationary point at $x = p$ in the interval $0 < x < \pi$.

i) Show that p satisfies the equation $\tan\frac{p}{2} = \frac{4}{p}$ [3]

ii) Verify by calculation that p lies between 2 and 2.5 [2]

iii) Use the iterative formula $p_{n+1} = 2\tan^{-1}\left(\frac{4}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(S-16 / 33 / Q6)

Q6. i) By sketching a suitable pair of graphs , show that the equation,

$\operatorname{cosec} \frac{x}{2} = \frac{x}{3} + 1$ has root in the interval $0 < x \leq \pi$ [2]

ii) Show by calculation that this root lies between 1.4 and 1.6 [2]

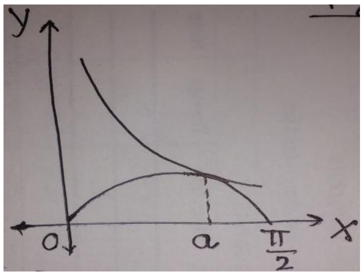
iii) Show that , if a sequence of values in the interval $0 < x \leq \pi$ given by the iterative formula $x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$ converges , then it converges to the root of the equation in part (i). [2]

iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(W-16 / 31/32 / Q6)

Q7. The diagram shows the curves, $y = x\cos x$ and $y = \frac{k}{x}$, where k is a constant , for $0 < x \leq \frac{\pi}{2}$

The curves touch at the point where $x = a$.



i) Show that **a** satisfies the equation $\tan a = \frac{2}{a}$ [5]

ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$ to determine **a** correct 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

iii) Hence the value of k correct to 2 decimal places. [2]

(W-16/33/Q9)

Q8. A curve has parametric equations :

$$x = t^2 + 3t + 1 \qquad , \qquad y = t^4 + 1$$

The point P on the curve has parameter p. It is given that the gradient of the curve at P is 4.

i) Show that $p = \sqrt[3]{(2p + 3)}$ [3]

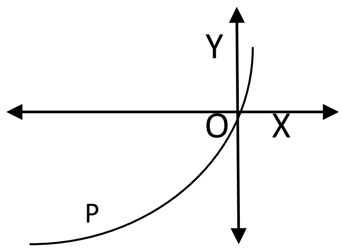
ii) Verify by calculation that the value of p lies between 1.8 and 2.0 [2]

iii) Use an iterative formula based on the equation in part (i) to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(W-15 / 33 / Q4)

Q9. The diagram shows the part of the curve with parametric equations :

$$x = 2 \ln (t + 2), \quad y = t^3 + 2t + 3$$



i) Find the gradient of the curve at the origin. [5]

ii) At the point P on the curve , the value of the parameter p .

It is given that the gradient of the curve at P is $\frac{1}{2}$.

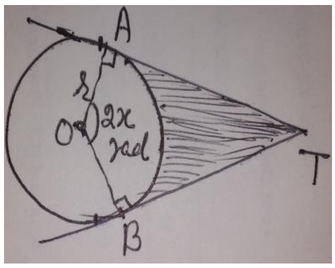
a) Show that $p = \frac{1}{3p^2 + 2} - 2$ [1]

b) By first using an iterative formula based on the equation in part (a) , determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and coordinate of P, correct to 2 decimal places. [4]

(S-15/ 31/ Q10)

Q10. The diagram show a circle with centre O and radius r. The tangents to the circle at point A and B meet at T, and angle AOB is 2x radians. The shaded region is bounded by the tangents AT and BT, and the minor arc AB. The perimeter of the shaded region is equal to the circumference of the circle.

i) Show that x satisfies the equation : $\tan x = \pi - x$ [3]



ii) This equation has one root in the interval $0 < x < \frac{\pi}{2}$.
Verify by calculation that this root lies between 1 and 1.3. [2]

iii) Use the iterative formula :

$$x_{n+1} = \tan^{-1} (\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(S-15 / 32 / Q5)

Q11. It is given that $\int_0^a x\cos x\,dx = 0.5$, where $0 < a < \frac{\pi}{2}$

i) Show that a satisfies the equation $\sin a = \frac{1.5 - \cos a}{a}$ [4]

ii) Verify by calculation that a is greater than 1. [2]

iii) Use the iterative formula :

$$a_{n+1} = \sin^{-1}\left(\frac{1.5 - \cos a_n}{a_n}\right)$$

to determine the value of **a** correct to 4 decimal places, giving the result of each iteration to 6 decimal places. [3]

(S-15 / 33 /Q6)

Q12. i) By sketching each of the graph $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation : $\operatorname{cosec} x = x(\pi - x)$ has exactly two real roots in the interval $0 < x < \pi$. [3]

ii) Show that the equation $\operatorname{cosec} x = (\pi - x)$ can be written in the form

$$x = \frac{1 + x^2 \sin x}{\pi \sin x}$$ [2]

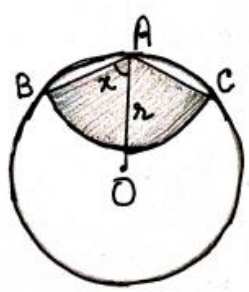
iii) The two real roots of the equation $\operatorname{cosec} x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β where $\alpha < \beta$

a) Use the iterative formula , $x_{n+1} = = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$ to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

b) Deduce the value of β correct to 2 decimal places. [1]

(S-14/ 31 / Q8)

Q13. In the diagram , A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is x radians. The shaded region is bounded by AB , AC and the circular arc with centre A joining B and C . The perimeter of the shaded region is equal to half the circumference of the circle.



i) Show that $x = \cos^{-1}\left(\frac{\pi}{4 + 4x}\right)$ [3]

ii) Verify by calculation that x lies between 1 and 1.5

iii) Use iterative formula : , $x_{n+1} = \cos^{-1}\left(\frac{\pi}{4+4x_n}\right)$

to determine the value of x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(S-14 / 32 / Q6)

Q14. The equation $x = \frac{10}{e^{2x}-1}$ has one positive root denoted by α.

i) Show that α lies between x = 1 and x =2. [2]

ii) Show that if a sequence of positive values given by the iteration formula:

$x_{n+1} = \frac{1}{2}\ln\left(1+\frac{10}{x_n}\right)$ converges , then it converges to α. [2]

iii) Use this iterative formula to determine α correct to 2 decimal places . Give the result of each iteration to 4 decimal places. [3]

(S-14 / 33 / Q4)

Q15. It is given that $\int_1^a \ln(2x)dx = 1$, where a> 1.

i) Show that $a = \frac{1}{2}\exp\left(1+\frac{\ln a}{2}\right)$, where exp(x) denotes e^x . [6]

ii) Use iterative formula : $a_{n+1} = \frac{1}{2}\exp\left(1+\frac{\ln a}{2}\right)$ to determine the value of **a** correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(W -14 / 31/ 32 / Q6)

Q16. i) Sketch the curve y = ln (x+1) and hence by sketching a second curve , show that the equation : $x^3 + \ln(x+1) = 40$ has exactly one real root. State the equation of the second curve. [3]

ii) Verify by calculation that the root lies between 3 and 4. [2]

iii) Use the iterative formula : $x_{n+1} = \sqrt[3]{(40-\ln(x_n+1))}$ with a suitable starting value , to find the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

iv) Deduce the root of the equation : $(e^y - 1)^3 + y = 40$, giving the answer correct to 2 decimal places. [2]

(W-14/33/Q9)

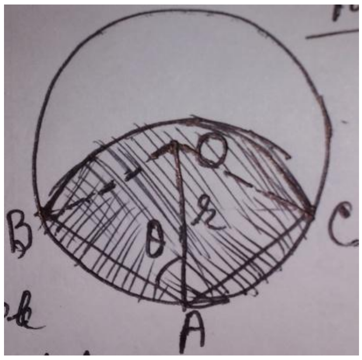
Q 17. In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is θ radians. The shaded region is bounded by the circumference on the circle and the arc with centre A joining B and C. The area of the shaded region is equal to half the area of the circle.

i) Show that $\cos 2\theta = \frac{2\sin 2\theta - \pi}{4\theta}$ [5]

ii) Use the iterative formula : $\theta_{n+1} = \frac{1}{2}\cos^{-1}\left(\frac{2\sin 2\theta_n - \pi}{4\theta_n}\right)$

with initial value $\theta_1 = 1$, to determine θ to 2 decimal places,

showing the result of each iteration to 4 decimal places.



[3]

(W -13 /31/32 / Q6)

Q18. It is given that $\int_0^p 4\, e^{\frac{-1x}{2}}\, dx = 9$, where p is a positive constant.

i) Show that $p = 2\ln\left(\frac{8p+16}{7}\right)$ [5]

ii) Use an iterative process based on the equation in part (i) to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. [3]

(W-13 / 33 / Q5)

Q19. Liquid is flowing into a small tank which has a leak. Initially the tank is empty and , t minutes later , the volume of liquid in the tank is V cm^3. The liquid is flowing into the tank at a constant rate of 80 cm^3 per minute. Because of the leak, liquid is being lost from the tank at a rate which , at any instant , is equal to kV cm^3 per minute, where k is a positive constant.

i) Write down a differential equation describing this situation and solve it to show that : $V = \frac{1}{k}(80 - 80\,e^{-kt})$ [7]

ii) It is observed that V = 500 when t = 15 , so that k satisfies the equation

$$k = \frac{4 - e^{-15k}}{25}$$

use the iterative formula based on this equation, to find the value of k correct to 2 significant figures. Use an initial value of k = 0.1 and show the result of each iteration to 4 significant figures. [3]

iii) Determine how much liquid is in the tank 20 minutes after the liquid started flowing, and state what happens to the volume of liquid in the tank after long time. [2]

(S -13 /31/Q10)

Q20. The sequence of values given by the iterative formula:

$$x_{n+1} = \frac{x_n\,(x_n^3 + 100)}{2\,(x_n^3 + 25)}$$

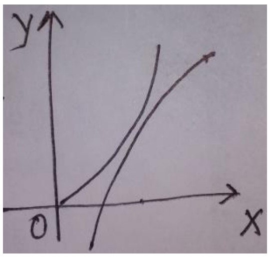
with initial value $x_1 = 3.5$, converges to α.

i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

ii) State an equation satisfied by α and hence find an exact value of α. [2]

(S -13 / 32/Q2)

Q21. The diagram shows the curves $y = e^{2x-3}$ and $y = 2\ln x$, when x =a the tangents to the curves are parallel.



i) Show that a satisfy the equation : $a = \frac{1}{2}(3 - \ln a)$ [3]

ii) Verify by calculation that this equation has a root between 1 and 2. [2]

iii) Use the iteration formula $a_{n+1} = \frac{1}{2}(3 - \ln a_n)$ to calculate **a** correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [2]

(S-13 /33/Q6)

Q22. i) It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting tan x by t , form an equation in t and hence show that either t = 0 or $t = \sqrt[3]{(t + 0.8)}$. [4]

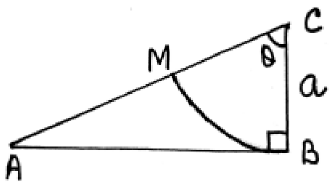
ii) It is given that there is exactly one real value of t satisfying the equation t = $\sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

iii) Use the iterative formula $t_{n+1} = \sqrt[3]{(t + 0.8)}$ to find the value of t correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

iv) Using the value of t found in previous parts of the equation , solve the equation : $2\tan 2x + 5\tan^2 x = 0$ for $-\pi \leq x \leq \pi$ [3]

(S-12 / 31 / Q10)

Q23. In diagram , ABC is a triangle in which angle ABC is right angle and BC = a. A circular arc , with centre C and radius a , joins B and the point M on AC. The angle ACB is θ radians. The area of the sector CMB is equal to c triangle ABC.



i) Show that θ satisfies the equation : $\tan\theta = 3\theta$ [2]

ii) This equation has one root in the interval $0 < \theta < \frac{\pi}{2}$. Use formula : $\theta_{n+1} = \tan^{-1}(3\theta_n)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

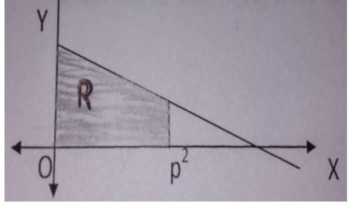
(S-12 / 32 / Q2)

Q24. The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x > 0$, where x is in radians. The shaded region between curve, the axes and the line $x = p^2$, where $p > 0$, is denotes by R . The area of R is equal to 1.

i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\, dx$. Hence show that

$\sin p = \frac{3 - 2\cos p}{2p}$ [6]

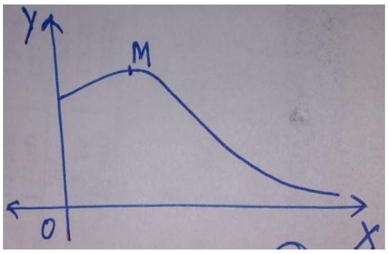
ii) Use the iterative formula $p_{n+1} = \sin^{-1}\left(\frac{3 - 2\cos p_n}{2p_n}\right)$ with initial value $p_1 = 1$, to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]



(S-12 / 33 / Q7)

Q25. The diagram shows the curve :

$$Y = e^{\frac{-1x^2}{2}} \sqrt{(1 + 2x^2)} \text{ for } x \geq 0 \text{, and its maximum point M.}$$



i) Find the exact value of the x – coordinate of M.

[4]

ii) The sequence of values given by the iterative formula :

$$x_{n+1} = \sqrt{\ln(4 + 8x_n^2)},$$

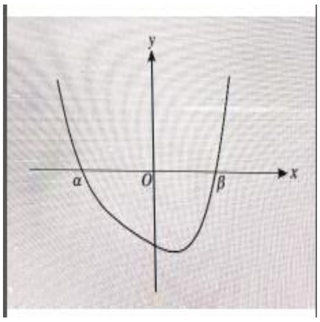
with initial value $x_1 = 2$, converges to a certain value α. State an equation satisfied by α and hence show that α is the x – coordinate of a point on the curve where y = 0.5 [3]

iii) Use the iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(W-12 / 31/ 32/Q8)

Q26. The diagram shows the curve :

$y = x^4 + 2x^3 + 2x^2 - 4x - 16$, which crosses the x- axis at the points (α , 0) and (β, 0) where α < β . It is given that α is an integer.



i) Find the value of α. [2]

ii) Show that β satisfies the equation $x = \sqrt[3]{(8 - 2x)}$ [3]

iii) Use an iteration process based on the equation in part (ii) to find the value of β correct to 2 decimal places. Show the result of each iteration to 4 decimal places. [3]

(W-12 / 33 / Q6)

A – Level - Mathematics **P_3** **<u>ITERATION (ANSWERS)</u>**

Q1. i) α lies between 2 and 3 ii) Prove iii) $\alpha = 2.219$ (Interation ---- 2.21877)	Q2. iii) $x = 1.78$ $\begin{cases} 1.77528 \\ 1.77528 \end{cases}$
Q3. iii) x 1.43 $\{$ 1.4304	Q4. iv) $a = 1.317$
Q5. iii) $p = 2.15$	Q6. iv) root $= 1.471$
Q7. $a = 1.077$ and $K = 0.55$	Q8. $p = 1.89$
Q9. i) $m = \frac{5}{2}$ at origin ii) (b) $p = -1.924$ and $P(-5.15, -7.97)$	Q10. iii) $x = 1.11$
Q11. $a = 1.2461$	Q12. iii) (a) $\alpha = 0.66$ (b) $\beta = 2.48$
Q13. iii) $x = 1.21$	Q14. iii) $\alpha = 1.14$
Q15. ii) $a = 1.94$	Q16. i) second curve : $y = 40 - x^3$ for $x > 0$ iii) $x = 3.377$ iv) 1.48
Q17. $\theta = 0.95$	Q18. ii) $p = 3.77$
Q19. ii) $K = 0.14$ iii) $530 \le V \le 540 \text{ cm}^3$ After long time $V = 569 \text{ cm}^3$	Q20. i) $x = 3.6840$ ii) $x = \frac{x(x^3 + 100)}{2(x^3 + 25)}$ $\therefore$ exact value of $\alpha = 3\sqrt{50}$
Q21. iii) $a = 1.35$	Q22. iii) $t = 1.276$ iv) -2.24 and 0.906 also $-\pi$, 0 , π
Q23. ii) $\theta = 1.32$	Q24. ii) $p = 1.25$
Q25. i) $x = \frac{1}{\sqrt{2}}$ ii) $\alpha = 1.86$	Q26. i) $\alpha = -2$ ii) $\beta = 1.67$

Application of an Interative Formula

Q8. iii) $p_{n+1} = \sqrt[3]{(2p_n + 3)}$

$p_0 = 1.8$

n	p_n	n	p_n
0	1.8	4	1.89317
1	1.87577	5	1.89326
2	1.89002	6	1.89328
3	1.89268	7	1.89328

Repeats

ANSWER :- p = 1.89 (upto 2 decimal places)

Calculator

SET

p_0 = 1.8	Exe.
	1.8
$\sqrt[3]{(2Ans.+3)}$	1.87577
	1.89002
	1.89268