Pure Maths 3

Iteration
Numerical Solution of Equations
Revision.

SP-20/M-20/S-20/M-19/S-19/W-19

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Example 1: The parametric equations of a curve are:
\[ x = e^{2t^3}, \quad y = 4 \ln t, \quad \text{where} \quad t > 0. \quad \text{When} \quad t = a, \quad \text{the gradient of curve is} \quad a. \]

(a) Show that \( a \) satisfies the equation \( a = \frac{1}{2} (3 - \ln a) \) \( \quad \) \( \text{[4]} \)

(b) Verify by calculation that this equation has a root between 1 and 2. \( \text{[3]} \)

(c) Use the iterative formula \( a_{n+1} = \frac{1}{2} (3 - \ln a_n) \) to calculate a correct to 2 d.p., showing the result of each iteration to 4 d.p. \( \text{[3]} \)

Solution:

\[ \frac{dx}{dt} = 6e^{2t^3}, \quad \frac{dy}{dt} = \frac{4}{t} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{4}{t}}{6e^{2t^3}} = \frac{2}{3te^{2t^3}} \]

\[ \text{Gradient at} \quad t = a \equiv \frac{2}{3ae^{2a^3}} = a \]

\[ \frac{dy}{dx} \bigg|_{t=a} = \frac{2}{3ae^{2a^3}} = a \]

\[ a = \frac{2}{3ae^{2a^3}} \Rightarrow a = e^{(3-2a)} \]

\[ \ln a = 3 - 2a \]

\[ a = \frac{1}{2} (3 - \ln a) \]

\[ a = \frac{1}{2} (3 - \ln a) \]

(b) \( a = \frac{1}{2} (3 - \ln a) \) \( \quad \) \( \text{[0]} \)

Consider \( f(a) = a - \frac{1}{2} (3 - \ln a) \)

at \( a = 1 \) \( f(1) = 1 - \frac{1}{2} (3 - 0) \)

and \( f(2) = 2 - \frac{1}{2} (3 - 0) \)

at \( a = 2; f(2) = 2 - \frac{1}{2} (3 - \ln 2) \)

\[ = 2 - \frac{1}{2} \times 2.30 \]

\[ = 2 - 1.15 = 0.85 \]

\[ \therefore f(a) \text{ changes sign between } a = 1 \text{ and } a = 2 \]

\( \therefore \) the equation \( \) has a root between 1 and 2.
Example 2 (a) By sketching a suitable pair of graph, show that the equation \( \sec x = 2 - \frac{1}{2}x \) has exactly one root in the interval \( 0 \leq x \leq \frac{\pi}{2} \).

(b) Verify by calculation that this root lies between 0.8 and 1.

(c) Use the iterative formula \( x_{n+1} = \cos^{-1} \left( \frac{3}{4-x_n} \right) \) to determine the root correct to 2 d.p. Give your result at each iteration to 4 d.p.

\[ x_0 = 0.8 \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( x_{n+1} = \cos^{-1} \left( \frac{3}{4-x_n} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.8956</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.8767</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.8756</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.8761</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.8759</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.8760</td>
<td></td>
</tr>
</tbody>
</table>

\[ x = 0.8760 \]
Example 3: The diagram shows a circle with centre O and radius r. The tangent to the circle at the point A and B meet at T, and angle AOB is 2x radians. The shaded region is bounded by the tangents AT and BT, and by minor arc AB. The area of the shaded region is equal to the area of the circle.

(A) Show that x satisfies the equation \( \tan x = \frac{\pi}{2} + x \)  \[-3\]

(B) This equation has one root in the interval \( 0 < x < \frac{\pi}{2} \). Verify by calculation that this root lies between 1 and 1.4.  \[-27\]

(C) Use iterative formula \( x_{n+1} = \tan^{-1} (\pi + x_n) \), to determine the root correct to 3 d.p. Give the result of each iteration to 4 d.p.  \[-6\]

\[ \tan AOT \]

Solution: \( AT = \tan x \)

\[(a) \quad \text{Area of } \triangle AOT = \frac{1}{2} \times \text{AT} \times \text{BT} \]

\[(b) \quad \text{Area of quadrilateral } OBAT = \text{Area of } \triangle AOB + \text{Area of } \triangle BTA \]

\[
\text{Area of sector } AOB = \frac{1}{2} \times r^2 \times \pi = \frac{r^2}{2} \times \pi
\]

\[
\text{Area of shaded region } = (\frac{r^2}{2} \times \tan x - \frac{r^2}{2} \times x)
\]

\[
\text{Give shaded area } = \text{area of circle}
\]

\[
\Rightarrow \frac{r^2}{2} \times (\tan x - x) = \pi r^2
\]

\[
\Rightarrow \tan x - x = \pi \Rightarrow \tan x = \pi + x
\]

\[ \tan x = \pi + x \quad \Rightarrow \]

\[
\text{Consider } f(x) = \tan x - \pi - x
\]

\[
f(1) = \tan 1 - \pi - 1 = -3.58\]

\[
f(1.4) = \tan 1.4 - \pi - 1.4 = 1.55
\]

\[
f(x) \text{ changes sign for } x = 1 \text{ and } 1.4
\]

\[ \Rightarrow \text{has a root between } 1 \text{ and } 1.4 \]

\[ x = 1.35 \quad \text{to } 2 \text{ d.p.} \]
Example: The diagram shows the curves

\[ y = \cos x \] and \[ y = \frac{k}{1 + x} \], where \( k \) is a constant, for \( 0 < x < \frac{\pi}{2} \).

The curves touch at the point where \( x = \pi \).

(a) Show that the \( \pi \) satisfies the equation,

\[ \tan \pi = \frac{1}{1 + \pi} \] \( \text{[5]} \)

(b) Use the iterative formula \( p_{n+1} = \tan^{-1} \left( \frac{1}{1 + p_n} \right) \) to determine the value of \( \pi \) correct to 3 d.p. Give the results of each iteration to 5 d.p.

8.20/33.689 \( \text{[8]} \)

Solution: \( y = \cos x \) \( \text{(1)} \)

(a) \( y = \frac{k}{1 + x} \) \( \text{(2)} \)

Both the curves \( (1) \) and \( (2) \) meet at \( x = \pi \).

\[ \cos \pi = \frac{k}{1 + \pi} \] \( \text{(3)} \)

From \( (1) \) diff. \( \frac{dy}{dx} = -\sin x \Rightarrow (dy)_x = -\sin \pi \]

for \( (2) \) \( \frac{dy}{dx} = -\frac{k}{(1 + x)^2} \Rightarrow (dy)_x = -\frac{k}{(1 + \pi)^2} \)

both the curves touch.

\[ \Rightarrow \sin \pi = f(\pi) \] \( \text{(3)} \)

\[ \Rightarrow \sin \pi = \frac{(1 + \pi) f(\pi)}{(1 + \pi)^2} \] \( \text{(4)} \)

\[ \Rightarrow \frac{1}{1 + \pi} \]

\[ \Rightarrow \tan \pi = \frac{1}{1 + \pi} \]

\[ \therefore x = 0.56778 \]

\[ = 0.568 \text{ to 3 d.p.} \]
Example 5 (a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root.

(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of equation in part (a).

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 d.p. Give the result of each iteration to 5 d.p.

\[8.2033206\] 

Solution: $y = 2 + x$ \(\text{and} \ y = x^5\)

(a) $y = x^5 \quad \text{(2)}\)

The graphs of eqns \(1\) and \(2\) intersect at only one point $P$, hence eqn $x^5 = 2 + x$ has exactly one root.

(b) Now consider the equation

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

\[x (5x^4 - 1) = 4x^5 + 2\]

\[5x^5 - x = 4x^5 + 2\]

\[x^5 = x + 2 \quad \text{is the equation in part (a),}\]

(c) $x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$

Now let $x_1 = 1.5$

$\begin{array}{|c|}
\hline
n & x_n \\
\hline
1 & 1.331619 \\
2 & 1.273516 \\
3 & 1.267236 \\
4 & 1.2671683 \\
\hline
\end{array}$

$x = 1.267$ \(\checkmark\)
Example 6: The sequence of values given by the iterative formula,

\[ x_{n+1} = \frac{2x_n^6 + 12x_n}{3x_n^5 + 8} \]

with initial value \( x_1 = 2 \), converges to \( x \).

(i) Use the formula to calculate \( x \) correct to 4 d.p. and give the result of each iteration to 6 decimal places. 

(ii) State an equation satisfied by \( x \) and hence find the exact value of \( x \). 

Solution:

\[ x_1 = 2 \]

\[ x_2 = 1.461538 \]

\[ x_3 = 1.322262 \]

\[ x_4 = 1.319507 \] \( \checkmark \)

\[ x_5 = 1.319507 \]

\[ \therefore x = 1.3195 \]

(ii) Now \( x = \frac{2x^6 + 12x}{3x^5 + 8} \)

\[ x(3x^5 + 8) = 2x^6 + 12x \]

\[ 3x^6 + 8x = 2x^6 + 12x \]

\[ x^6 - 4x = 0 \]

\[ x(x^5 - 4) = 0 \]

\[ x^5 - 4 = 0 \]

\[ x^5 = 4 \]

\[ x = \frac{\sqrt[5]{4}}{} \]
Example 7: The diagram shows the curves
\[ y = 4\cos \frac{1}{2}x \quad \text{and} \quad y = \frac{1}{4-x} \quad \text{for} \quad 0 \leq x < 4, \]
when \( x = a \), the tangents to the curves
are perpendicular.

(i) Show that \( a = 4 - \sqrt{2\sin \frac{1}{4}a} \)

(ii) Verify by calculation that \( a \) lies between 2 and 3.

(iii) Use an iterative formula based on the equation in part (1) to
determine a correct to 3 decimal places, and the result of each
iteration to 5 decimal places.

Solution: \( y = 4 \cos \frac{1}{2}x \)  

(i)
\[ \frac{dy}{dx} = -4 \sin \frac{1}{2}x \quad (\text{at } x = a) \]
\[ \frac{1}{4-x} \]

and \( y = (4-x)^{-1} \)

(ii)
\[ \frac{dy}{dx} = \frac{1}{(4-x)^2} \]
\[ (\text{at } x = a) \]

Tangents to (1) and (2) are perpendicular.

The curve at \( x = a \) \( \left[ m_1m_2 = -1 \right] \)

\[ -2 \sin \frac{1}{4}a \times \frac{1}{(4-a)^2} = -1 \]
\[ \Rightarrow 2\sin \frac{1}{4}a = (4-a)^2 \]
\[ \Rightarrow 4-a = \sqrt{2\sin \frac{1}{4}a} \]
\[ \Rightarrow a = 4 - \sqrt{2\sin \frac{1}{4}a} \]

(iii) Use iteration formula

\[ a_1 = 2.5 \]
\[ a_{n+1} = 4 - \sqrt{2 \sin \frac{1}{4}a} \]

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.62233</td>
</tr>
<tr>
<td>3</td>
<td>2.60968</td>
</tr>
<tr>
<td>4</td>
<td>2.61086</td>
</tr>
<tr>
<td>5</td>
<td>2.61075</td>
</tr>
<tr>
<td>6</td>
<td>2.61076</td>
</tr>
</tbody>
</table>

\( a = 2.61076 \) \( \text{upto 3 d.p.} \)

\( a = 2.611 \) \( \text{upto 3 d.p.} \)

\( a \) lies between 2 and 3.
Example 8: In the diagram, A is the mid-point of the semicircle with centre O and radius r. A circular arc with centre A meets the semicircle at B and C. The angle OAB is equal to x radians. The area of the shaded region bounded by AB, AC and the arc with centre A is equal to half the area of the semicircle.

(i) Use triangle OAB, to show \( AB = 2r \cos x \)  \(-1\)

(ii) Hence show that \( x = \cos^{-1} \left( \frac{\pi}{16x} \right) \) \(-3\)

(iii) Verify by calculation that x lies between 1 and 1.5 \(-3\)

(iv) Use an iterative formula based on the equation in part (ii) to determine x correct to 3 d.p. Give the result of each iteration to 5 d.p.

Solution: (i) In isosceles \( \triangle OAB \)

\[ OA = OB = r \]

Arrow \( ON \perp AB \)

\[ AN = AB \]

In right \( \triangle OAN, \quad AN = \cos x \]

\[ \Rightarrow \frac{AB}{\cos x} = OA \]

\[ \Rightarrow AB = 2r \cos x \quad (1) \]

(ii) Area of shaded region = \( \frac{1}{2} (AB)^2 \times 2x \)

\[ \text{From } (1) \]

\[ = 4r^2 \cos^2 x \quad (2) \]

\[ \text{Given shaded area} = \frac{1}{2} \text{ area of semicircle} \]

\[ \text{From } (2) \]

\[ 4r^2 \cos^2 x = \frac{1}{2} \left( \frac{\pi}{2} \right) \]

\[ \Rightarrow \cos^2 x = \frac{\pi}{16x} \]

\[ \Rightarrow \cos x = \sqrt{\left( \frac{\pi}{16x} \right)} \Rightarrow x = \cos^{-1} \left( \frac{\pi}{16x} \right) \]

\[ \text{or } x = 1.14354 \text{ \text{upto 3 d.p.}} \]
Example 9: The diagram shows the curve 

\[ y = x^4 - 2x^3 + 7x - 6 \]. The curve intersects the \( x \)-axis at the points \((a,0)\) and \((b,0)\) where \( a < b \). It is given that \( b \) is an integer.

(i) Find the value of \( b \).

(ii) Show that \( a \) satisfies the equation

\[ a = -\frac{1}{3} (2 + a^2 + a^3) \]

(iii) Use an iterative formula based on the equation in part (ii) to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

\[ \text{Solution: } \text{df}(x) = 4x^3 - 6x^2 - 7x - 6 \]

(i) \( a = 2 \), \( f(2) = 16 - 16 - 14 - 6 = -26 \neq 0 \)

\( x = 3 \), \( f(3) = 81 - 54 - 21 - 6 = 0 \)

\( \therefore \text{using factor theorem, } b = 3 \)

(ii) \((x-3)\) is a factor of \( f(x) \) divided by \((x-3)\)

\((x-3)[x^3 + x^2 + 3x + 2] = 0 \)

\( a^3 + a^2 + 3a + 2 = 0 \)

\( \Rightarrow 3a = -(a^3 + a^2 + 2) \)

\( \Rightarrow a = -\frac{1}{3} (2 + a^2 + a^3) \)

(iii) Using iterative formula

\begin{align*}
\text{Let } a_1 &= 1 \\
\text{Then } a_{n+1} &= -\frac{1}{3} (2 + a_n^2 + a_n^3)
\end{align*}

\begin{align*}
a_2 &= -1.33333 \\
a_3 &= -0.46913 \\
a_4 &= -0.70561 \\
a_5 &= -0.71522 \\
a_6 &= -0.71521 \\
a_7 &= -0.71522 \checkmark \\
a_8 &= -0.71522 \\
\therefore a = -0.715 \checkmark
\end{align*}
Example 10: The curve with equation \( y = e^{2x} \ln(x-1) \) has a stationary point when \( x = p \).

(i) Show that \( p \) satisfies the equation \( x = 1 + \exp\left(\frac{1}{2(x-1)}\right) \), where \( \exp(x) \) denotes \( e^x \). \([3]\)

(ii) Verify by calculation that \( p \) lies between 2.2 and 2.6. \([2]\)

(iii) Use an iterative formula on the equation in part (i) to determine \( p \) correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \([W-10\, 31\, 025]\) \([3]\)

Solution: \( y = e^{2x} \ln(x-1) \)

(i) \( \frac{dy}{dx} = -2e^{2x} \ln(x-1) + \frac{e^{2x}}{x-1} \)

For stationary point \( \frac{dy}{dx} = 0 \)

\( -2e^{2x} \ln(x-1) + \frac{e^{2x}}{x-1} = 0 \)

\( e^{2x} \left[ \frac{1}{x-1} - 2 \ln(x-1) \right] = 0 \)

\( 2 \ln(x-1) = \frac{1}{x-1} \)

\( \ln(x-1) = \frac{1}{2(x-1)} \)

or \( x-1 = \exp\left(\frac{1}{2(x-1)}\right) \)

\( p \) satisfies the eqn:

\( x = 1 + \exp\left(\frac{1}{2(x-1)}\right) \)

\( \therefore \) Taking \( p_{n+1} = 1 + \exp\left(\frac{1}{2(p_n-1)}\right) \)

\( p_1 = 2.4 \)

\( p_2 = 2.4292 \)

\( 2.4218 \)

\( p_3 = 2.4215 \)

\( p_4 = 2.4215 \) \( \checkmark \)

(ii) Consider \( f(x) = x-1 - e^{\left(\frac{1}{2(x-1)}\right)} \)

\( f(2.2) = 2.2 - 1 - e^{\left(\frac{1}{2(2.2-1)}\right)} \)

\( = -0.475 \) \( \checkmark \)

\( f(2.6) = 2.6 - 1 - e^{\left(\frac{1}{2(2.6-1)}\right)} \)

\( = 0.233 \) \( \checkmark \)

\( f(x) \) changes of sign between \( x = 2.2 \) and 2.6

\( \therefore p \) lies lies between 2.2 and 2.6
Example 11: It is given that \[ \int_0^a x \cos \frac{1}{2} x \, dx = 3, \] where the constant a is such that \( 0 < a < \frac{3\pi}{2} \).

(i) Show that a satisfies the equation \[ a = 4 - 3 \cos \frac{1}{3} a \] \[ \sin \frac{1}{3} a \] \[ -15 \]

(ii) Verify by calculation that a lies between 2.5 and 3.  

(iii) Use an iterative formula based on the equation in part (i) to calculate a correct to 5 decimal places. Give the result of each iteration to 5 decimal places. \[ [\text{W}-19/32/\text{Q9}] \] \[ -13 \]

Solution: \[ \int x \cos \frac{1}{2} x \, dx \]

(C) \[ = x \cdot \int \cos \frac{1}{2} x \, dx - \int \left( \frac{d}{dx} \cos \frac{1}{2} x \right) \, dx \]

\[ = 3x \cdot \sin \frac{1}{2} x - \int \left( 3 \cdot \frac{1}{2} \sin \frac{1}{2} x \right) \, dx \]

\[ = 3x \cdot \sin \frac{1}{2} x - \frac{3}{2} \left( -2 \cos \frac{1}{2} x \right) \]

\[ = 3x \sin \frac{1}{2} x + 9 \cos \frac{1}{2} x \]

\[ \Rightarrow \int_0^a x \cos \frac{1}{2} x \, dx = \left[ 3x \sin \frac{1}{2} x + 9 \cos \frac{1}{2} x \right]_0^a \]

\[ \Rightarrow 3a \sin \frac{1}{2} a + 9 \cos \frac{1}{2} a - 9 = 3 \sin a \]

\[ \Rightarrow a \sin \frac{1}{3} a + 3 \cos \frac{1}{3} a = 4 \]

\[ \Rightarrow a = 4 - 3 \cos \frac{1}{3} a \]

\[ \sin \frac{1}{3} a \]

(iii) Use the iterative formula

\[ a_{n+1} = 4 - 3 \cos \frac{1}{3} a_n \]

\[ a_1 = 2.7 \]

\[ a_2 = 2.72577 \]

\[ a_3 = 2.73309 \]

\[ a_4 = 2.73523 \]

\[ a_5 = 2.73585 \]

\[ a_6 = 2.73604 \]

\[ a_7 = 2.73609 \]

\[ a_8 = 2.73611 \]

\[ a_9 = 2.73611 \]

\[ \therefore a = 2.736 \]

(ii) Consider \( f(a) = 9 \sin \frac{1}{3} a + 3 \cos \frac{1}{3} a - 4 \)

\[ f(2.5) = 2.5 \sin \frac{1}{3} 2.5 + 3 \cos \frac{1}{3} 2.5 - 4 \]

\[ = -0.14 \checkmark \]

and \[ f(3) = 3 \sin \frac{1}{3} 3 + 3 \cos \frac{1}{3} 3 - 4 \]

\[ = 0.14 \checkmark \]

\[ f(a) \text{ changes sign for } 2.5 \text{ and } 3. \]

\[ \therefore a \text{ lies between } 2.5 \text{ and } 3. \checkmark \]
Example 12 (i) By sketching suitable pairs of graphs, show that the equation $\ln(x+2) = 4e^{-x}$ has exactly one real root. \[\text{--[2]}\]

(ii) Show by calculation that this root lies between $x = 1$ and $x = 1.5$. \[\text{--[2]}\]

(iii) Use the iterative formula $x_{n+1} = \ln\left(\frac{4}{\ln(x_n + 2)}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[\text{--[3]}\]

**Solution:** Consider $y = \ln(x+2)$ \[\text{(1)}\]

and $y = 4e^{-x}$ \[\text{(2)}\]

Graphs of (1) and (2) intersect exactly one point $P$. \[\checkmark\]

(ii) Consider $f(x) = \ln(x+2) - 4e^{-x}$

for $x = 1$, $f(1) = \ln 3 - 4e^{-1} = -0.381$

$x = 1.5$, $f(1.5) = \ln 3.5 - 4e^{-1.5} = 0.360$

$f(x)$ changes sign for $x = 1$ & $x = 1.5$

hence $x$ lies between 1 & 1.5. \[\checkmark\]

(iii) Use iterative formula.

<table>
<thead>
<tr>
<th>$x_{n+1}$ = \ln(\frac{4}{\ln(x_n + 2)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>let $x_1 = 1.2$</td>
</tr>
<tr>
<td>$x_2$ = 1.2355</td>
</tr>
<tr>
<td>1.2258</td>
</tr>
<tr>
<td>1.2276</td>
</tr>
<tr>
<td>1.2277 \checkmark</td>
</tr>
<tr>
<td>1.2277</td>
</tr>
</tbody>
</table>

$x = 1.2277$

\[\therefore x = 1.23 \checkmark\text{ correct to 2 d.p.}\]