(P3) Solving Equations Numerically [Notes] Page I

Iteration

1. Sign Change Rule: If the function $f(x)$ is continuous in some interval $p \leq x \leq q$ of its domain, and if $f(p)$ and $f(q)$ have opposite signs, then $f(x) = 0$ has atleast one root between p and q.

2. Decimal Search:

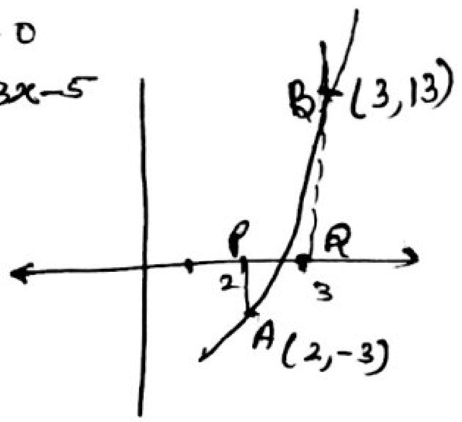
To solve the equation $x^3 - 3x - 5 = 0$

Consider $f(x) = x^3 - 3x - 5$

(i) $f(2) = -3$

$f(3) = 13$

It follows that required root lies between 2 & 3.



(ii) $f(2.2) = -0.952$

$f(2.3) = 0.267$

$\therefore$ root is between 2.2 and 2.3

(iii) $f(2.27) = -0.112$ and $f(2.28) = -0.112$

Proceeding similarly we can find the root $x = 2.279$ to 3 decimal places

[3] Finding roots by Iteration:

Example 1 To solve the equation $x^3 - 3x - 5 = 0$

This can be rearranged as $x = \sqrt[3]{3x+5}$

In general $x = F(x)$

now use a sequence defined as.

$$x_{r+1} = \sqrt[3]{3x_r + 5}$$

Continued

(P3) Solving Equations Numerically: **Notes**

Iteration method:

Example 1 continued

$$x_{r+1} = \sqrt[3]{3x_r + 5}$$

we can find that the root is close to 2 (using sign change rule)

$\therefore x_0 = 2.$

r	x_r	r	x_r	r	x_r
0	2	4	2.27862	8	2.27902
1	2.22398	5	2.27894	9	2.27902
2	2.26837	6	2.27900		
3	2.27697	7	2.27902		

$\therefore$ This suggests that $x = 2.27902$ to 5 decimal places

Calculator

Set-2 - - - - - - - - - - - - - - Enter

2

$\sqrt[3]{3\,\text{Ans} + 5}$ - - - - - - - - 2.223980091

2.268372388

continued until it repeats 3 times

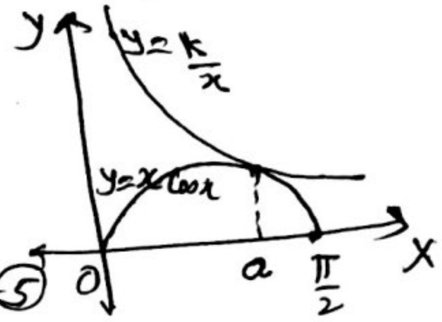
(P3) Solving Equations Numerically — **Notes** — Page-3

Iteration Method

Example 2:

(Q7). The diagram shows the curves $y = x\cos x$ and $y = \frac{k}{x}$ for $0 < x \le \frac{\pi}{2}$; k is a constant. **W-16/33/Q9**

The curves touch at the point where $x = a$



(i) Show that the curves touch at the point ~~where x =~~ 'a' that satisfy the equation $\tan a = \frac{2}{a}$ -- (5)

(ii) Use iterative formula $a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$ to determine 'a' correct to 3 decimal places. Give the result of of each iteration to 5 decimal places. -- (3)

(iii) Hence the value of k correct to 2 decimal places. - (2).

Solution (i) $y = x\cos x$ and $y = \frac{k}{x}$

$\frac{dy}{dx} = \cos x - x\sin x$ — (1) and $\frac{dy}{dx} = -\frac{k}{x^2}$ — (2)

$\left(\frac{dy}{dx}\right)_{x=a} = \cos a - a\sin a$ — (1)' & $\left(\frac{dy}{dx}\right)_{x=a} = -\frac{k}{a^2}$ — (2)'

The curves touch at $x = a$

$\therefore$ for (1)' and (2)' $\cos a - a\sin a = -\frac{k}{a^2}$ — (3)

Also equating the y the value of y at $x = a$ for both the curve

$a\cos a = \frac{k}{a} \Rightarrow k = a^2\cos a$ — (4)

for (3) & (4) $\cos a - a\sin a = -\frac{a^2\cos a}{a^2} \Rightarrow 2\cos a = a\sin a$

$\Rightarrow \tan a = \frac{2}{a}$ — (5)

Continue

Iteration Method

Example 2 (Q7) Continued: (ii)

for (3) $\tan a = \frac{2}{a} \Rightarrow a = \tan^{-1}\left(\frac{2}{a}\right)$

now set Iteration formula

$$a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$$

Now let $a_0 = 1$

r	a_r	r	a_r	r	a_r
0	1	5	1.077554	10	1.076868
1	1.10714	6	1.076610	11	1.076876
2	1.065213	7	1.076976	12	1.076873
3	1.081405	8	1.076834	13	1.076874
4	1.075119	9	1.076889	14	1.076873

$\therefore$ Req $a = 1.077$

(iii) for (4) $K = a^2 \cos a$

$= (1.077)^2 \cdot \cos 1.077$

$= 0.5497$

$K = 0.55$ (2 d.p)