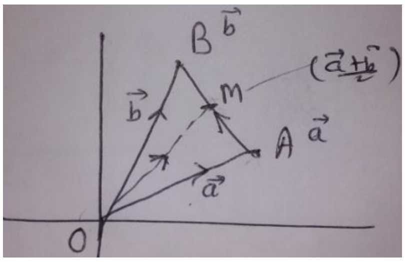
A- LEVEL – MATHEMATICS P_3 VECTORS IN 3D (Notes)

Position Vector of Points A , B are $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$



i) $\overrightarrow{AB} = (\vec{b} - \vec{a})$

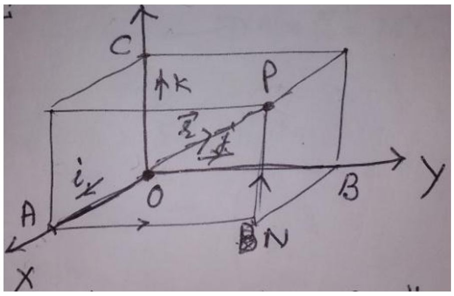
ii) Position Vector of the Mid point of AB , M

$$\overrightarrow{OM} = \frac{\vec{a} + \vec{b}}{2}$$

$$\because \overrightarrow{OM} = \vec{a} + \overrightarrow{AM} = \vec{a} + \frac{\overrightarrow{AB}}{2} = \vec{a} + \frac{(\vec{b} - \vec{a})}{2} = \frac{\vec{a} + \vec{b}}{2}$$

Components of Vectors in 3D :

Unit Vectors along the axes OX , OY, OZ are denoted by i , j , k respectively.



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AN} + \overrightarrow{NP}$

or $\overrightarrow{OP} = (x\,i + y\,j + z\,k)$

is the position vector of **variable point P** .

$\vec{\mathbf{r}}$ or $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where OA = x , AN = OB = y , NP = OC = z

Distance OP = $|\vec{\mathbf{r}}| = \sqrt{(x^2 + y^2 + z^2)}$

Position vectors of given points :

A (a_1 , a_2 , a_3) ; $\overrightarrow{OA} = a_1\,i + a_2\,j + a_3\,k = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \vec{\mathbf{a}}$

and B (b_1 , b_2 , b_3) ; $\overrightarrow{OB} = b_1\,i + b_2\,j + b_3\,k = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \vec{\mathbf{b}}$

and $\overrightarrow{AB} = (\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{a}}) = \begin{pmatrix} b_1 - & a_1 \\ b_2 - & a_2 \\ b_3 - & a_3 \end{pmatrix}$

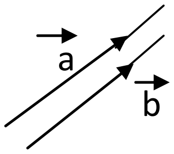
Magnitude of $\overrightarrow{a}$

$OA = |\overrightarrow{\mathbf{a}}| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$

Unit Vector along $\overrightarrow{\boldsymbol{a}} = \widehat{\boldsymbol{a}} = \dfrac{\overrightarrow{\boldsymbol{a}}}{|\overrightarrow{\boldsymbol{a}}|}$

$$= \frac{a_1}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}} i + \frac{a_2}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}} j + \frac{a_1}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}} k$$

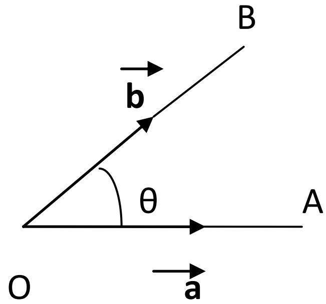
Parallel Vectors



or $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = k\vec{b}$, $k \in R$, $k \neq 0$

or = $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = k \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Leftrightarrow \vec{a} \parallel \vec{b}$

Scalar Product of Vectors



Def^n $\quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \cos\theta$

Or $\quad \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$ --------------------- (ii)

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors along axes (are mutually perpendicular)

$\mathbf{i} \cdot \mathbf{i} = \mathbf{i}^2 = 1 \times 1 \times \cos 0^0 = 1 = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k}$

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ --------------------- (iii) also $\vec{a} \cdot \vec{a} = (\vec{a})^2 = |\vec{\mathbf{a}}|^2$

and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$ -------------------- (iv)

and $\vec{\mathbf{a}} \perp \vec{\mathbf{b}} \Leftrightarrow \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{0}$, $\vec{\mathbf{a}} \neq \mathbf{0}$, $\vec{\mathbf{b}} \neq \mathbf{0}$

Now given $\vec{\mathbf{a}} = a_1 i + a_2 j + a_3 k = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{\mathbf{b}} = b_1 i + b_2 j + b_3 k = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

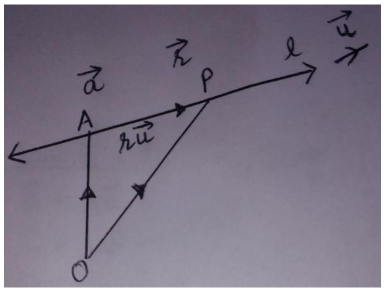
$\vec{a} \cdot \vec{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3)$ ---------------------------- (v)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)} \cdot \sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$ ------------------------- vi)

<u>Equation of a line</u> ' l ' passing through a point A whose position vector $\vec{a}$ and direction of line is $\vec{u}$

$\vec{r} = \vec{a} + \lambda \vec{\mu}$ --- (i) as $\vec{OP} = \vec{OA} + \vec{AP}$



$$\vec{a} = a_1 i + a_2 j + a_3 k = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\vec{r} = x i + y j + z k = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Director of line ' l '

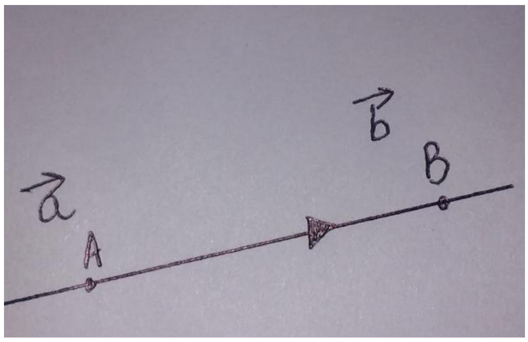
$$\vec{u} = p i + q j + r k = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Equation of line l

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix} \text{ ------------------- (ii)}$$

2. Equation of line passing through two points $\vec{a}$ **and** $\vec{b}$

$\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$ **--------- (iii)**



Direction $\vec{AB} = (\vec{b} - \vec{a})$

$$\vec{u} = \begin{pmatrix} b_1 - & a_1 \\ b_2 - & a_2 \\ b_3 - & a_3 \end{pmatrix}$$

3. <u>To verify that two given line l_1 and l_2</u> (May be PARALLEL / COINCIDENT / INTERSECTING / SKEW LINES) :

l_1 : $\vec{r} = \vec{a} + \lambda \vec{u}$ ------- **(i)** **where** $\vec{u} = p\,i + q\,j + r\,k = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

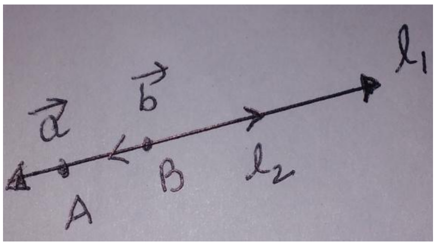
l_2 : $\vec{r} = \vec{b} + \lambda \vec{v}$ -------(ii) **and** $\vec{v} = l\,i + m\,j + n\,k = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

Case (a) : $l_1 \parallel l_2 \Longleftrightarrow \vec{u} = k\vec{v}$: $k \in R$, $k \neq 0$

Case (b) : $l_1 \parallel l_2$ are coincident lines if

i) $\vec{u} = k_1 \vec{v}$

(ii) $(\vec{b} - \vec{a}) = k_2 \vec{u}$



Case (c) : **Intersecting** $\vec{u} \neq k\vec{v}$; $l_1 \nparallel l_2$

To find the point of intersection l_1 : $\vec{r} = \begin{pmatrix} a_1 + \lambda p \\ a_2 + \lambda q \\ a_3 + \lambda r \end{pmatrix}$ ------------ (iii)

l_2 : $\vec{r} = \begin{pmatrix} b_1 + \mu l \\ b_2 + \mu m \\ b_3 + \mu n \end{pmatrix}$ --------------- (iv)

For a Common point :

$$\begin{pmatrix} a_1 + \lambda p \\ a_2 + \lambda q \\ a_3 + \lambda r \end{pmatrix} = \begin{pmatrix} b_1 + \mu l \\ b_2 + \mu m \\ b_3 + \mu n \end{pmatrix}$$

or $a_1 + \lambda p = b_1 + \mu l \Rightarrow \lambda p - \mu l = b_1 - a_1$ ------------ (v)

$a_2 + \lambda q = b_2 + \mu m \Rightarrow \lambda q - \mu m = b_2 - a_2$ ------------ (vi)

$a_3 + \lambda r = b_3 + \mu n \Rightarrow \lambda r - \mu n = b_3 - a_3$ ------------ (vii)

Solve (v) and (vi) for λ and μ

And verify that these values of λ and μ satisfies the equation (vii) ; and to **find the point of intersection**, put the value of λ in equation(iii) (or μ in (iv))

3. d) Pair of lines l_1 and l_2 are Skew :

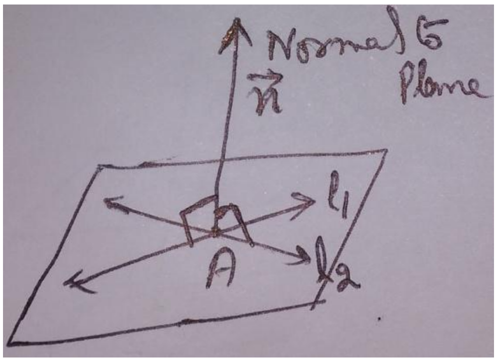
$l_1 \nparallel l_2$ and l_1 and l_2 are non intersecting.

It happens when in [3] (c) we solve two equations for λ and μ but these values of λ and μ does not satisfy the third equation.

PLANE IN 3D

Direction of a Plane is expressed in terms of its **Normal** $\vec{n}$ to the Plane :

Normal to the Plane is perpendicular to every line lying in the plane, through the point of intersection of Plane and normal.

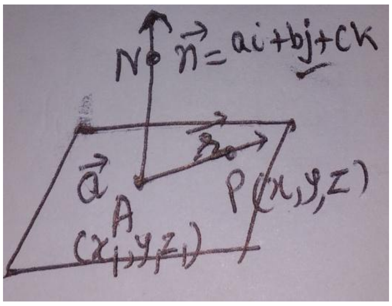


$\vec{n} \perp l_1$ and $\vec{n} \perp l_2$

1. **Vector Equation of a Plane :**

   i) Passing through a point $\vec{a}$ and given $\vec{n}$ is the normal to the plane , $\vec{r}$ is any point (variable) on the plane.

$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ ------(i) [∵ $\overrightarrow{AP} \perp$ Normal]



General Equation of Plane (Vector form)

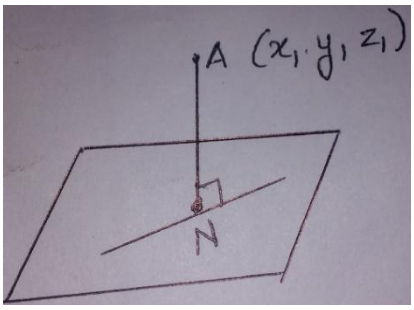
$\vec{r} \cdot \vec{n} = d$ --------- (ii)

2. **Cartesian Equation of a Plane :**

   i) Passing through a point A (x_1, y_1, z_1) and components of normal are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ----- (iii) [$\vec{n} = ai + bj + ck$]

General Equation of Plane in Cartesian form:



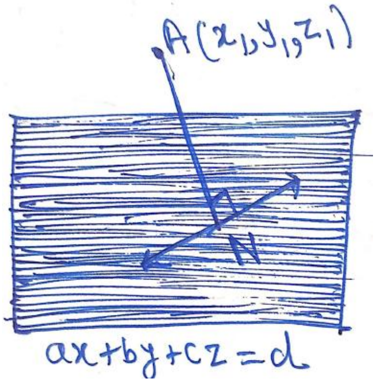
$ax + by + cz = d$ ----------------- (iv)

here a , b , c are Components of Normal

3. i) Length of perpendicular from a point to a Plane :

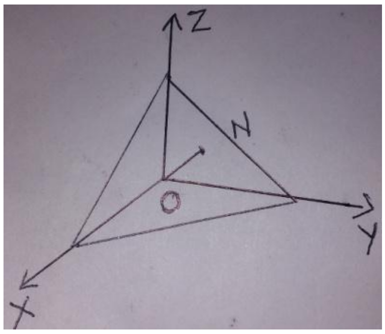
Given a point A (x_1, y_1, z_1)

and a plane $ax + by + cz = d$



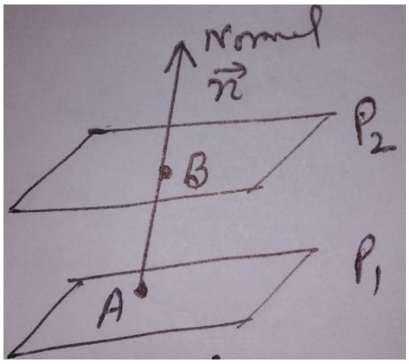
Length of Perpendicular **AN** = $\dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{(a^2 + b^2 + c^2)}}$

ii) Length of perpendicular from origin to the Plane :



$$ON = \frac{|d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

4. i) <u>**Parallel Planes**</u>



Two Planes are parallel iff they have the same normal .i.e either the components of normal are same or proportional.

P_1 : $a_1x + b_1y + c_1z = d_1$

P_2 : $a_2x + b_2y + c_2z = d_2$

$\vec{n_1} = a_1 i + b_1 j + c_1 k$

$\vec{n_2} = a_2 i + b_2 j + c_2 k$

$$P_1 \parallel P_2 \Rightarrow \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = k \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad : k \in R \text{ and } k \neq 0$$

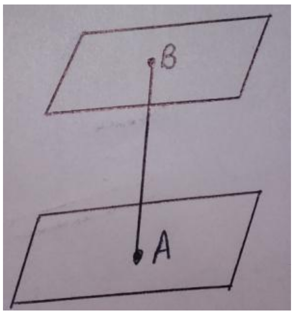
Parallel Planes

$$\begin{cases} 2x - 3y + z = 7 \\ 6x - 9y + 3z = 10 \end{cases} \quad \text{or} \quad \begin{cases} 3x - 5y + 2z = 6 \\ 3x - 5y + 2z = 9 \end{cases}$$

ii) Distance between two Parallel Planes

a) P_1 : $ax + by + cz = d_1$
P_2 : $ax + by + cz = d_2$

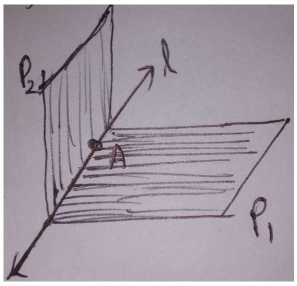
Make the coefficient of x , y , z in both the equations equal.



$$\text{Distance} \quad AB = \frac{|d_1 - d_2|}{\sqrt{(a^2 + b^2 + c^2)}}$$

b) Alternate Method : Take any point on plane P_1 and find the distance (length of perpendicular) of this point to second plane.

5. Equation of a Plane passing through the intersection of two given planes:



P_1 : $a_1x + b_1y + c_1z = d_1$
P_2 : $a_2x + b_2y + c_2z = d_2$

is given by :

$$(a_1x + b_1y + c_1z - d_1) + \lambda\,(a_2x + b_2y + c_2z - d_2) = 0$$

6. To find the equation of a plane passing through three points A (x_1, y_1, z_1) , B (x_2, y_2, z_2) , C (x_3, y_3, z_3)

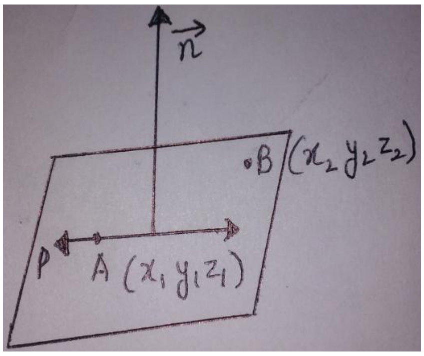
May be given :

$\vec{OA} = x_1 i + y_1 j + z_1 k$
$\vec{OB} = x_2 i + y_2 j + z_2 k$
$\vec{OC} = x_3 i + y_3 j + z_3 k$
Position Vector of A,B, C

Equation of any plane through point A (x_1, y_1, z_1) is
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ------------------ (i)
B (x_2, y_2, z_2) lies on (i) $\longrightarrow$ $a(x_2 - x_1)$ + ----+--- = 0 ---------(ii)
C (x_3, y_3, z_3) lies on (i) $\longrightarrow$ $a(x_3 - x_1)$ + ---+-- - = 0 ---------- (iii)

Solve (ii) and (iii) by cross –multiplication method and put the values of a, b , c in (i)

7. To find the Equation of Plane passing through line ' l ' and point B (x_2 , y_2 , z_2)



$$l : \vec{r} = \mathbf{a} + \lambda \vec{u}$$

or $l : \vec{r} = (x_1 i + y_1 j + z_1 k) + \lambda (p i + q j + r k)$

Now Point A (x_1 , y_1, z_1) on line 'l' lies on Plane

∴ *Equation of Plane through A* (x_1 , y_1, z_1)

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ------------------ (i)

and the given point B (x_2 , y_2, z_2) lies on required Plane

Put in (i)

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ ------------------ (ii)

as line ' l ' lies in plane.

$$l \perp \text{Normal}$$

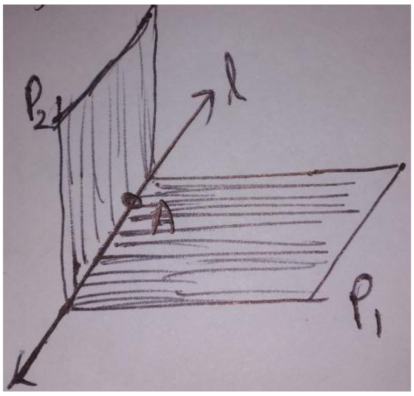
$$\vec{u} \cdot \vec{n} = 0$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$\Rightarrow a p + b q + c r = 0$ --------------------------------- (iii)

Solve equations (ii) and (iii) for a , b and c by cross multiplication and put the values of a , b ,c in (i)

8. To find the equation of the Line ' l ' of intersection of two planes:



Given Two Planes

$P_1 : a_1x + b_1y + c_1z = d_1$ ------------------ (i)

$P_2 : a_2x + b_2y + c_2z = d_2$ ------------------ (ii)

Put x = 0 in equation (i) and (ii) , we get

$b_1y + c_1z = d_1$
$b_2y + c_2z = d_2$

Solve for y and z

Get the coordinate of a common point A (0 , y_1 , z_1)
Again put y = 0 (or may z = 0) and get

$a_1x + c_1z = d_1$
$a_2x + c_2z = d_2$

Solve for x and z to get B (x_2 , 0 , z_2)

As A, B lies on Required line ' l '
Find the equation of line through two points A and B.

9. <u>To find the distance of a point B (x_2 , y_2 , z_2) from a line :</u>

Given B (x_2 , y_2 , z_2)

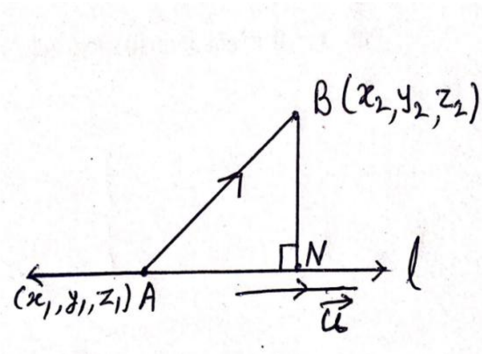
Line l : $\vec{r} = \vec{a} + \lambda \vec{u}$

Or $\vec{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Find $\vec{AB} = (x_2 - x_1)\, i + (y_2 - y_2)\, j + (z_2 - z_1)\, k$

Now AN = **Projection of $\vec{AB}$ on line l**

$= \vec{AB} \cdot \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$ $\left\{ \vec{\mathbf{u}} = p\ i + q\, j + r\, k \right\}$



Required length of perpendicular distance

BN = √ (AB^2 - AN^2)

GENERAL RESULTS :

i) Any line ∥ to x-axis has Direction $\vec{V} = a\,i = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ii) A Plane ∥ x-axis ⟹ Normal to Plane $\vec{n} \perp$ x- axis

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

⟹ a = 0

$$\therefore \vec{n} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

iii) Line l : $\vec{r} = \vec{a} + \lambda\,\vec{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Plane P : $\vec{r} \cdot \vec{n} = d$ ⟹ $ax + by + cz = d$

then **(a)** line l ∥ Plane P ⟹ l ⊥ normal

⟹ $\vec{\mathbf{u}} \cdot \vec{\mathbf{n}} = 0$

or $ap + bq + cr = 0$

(b) l ⊥ Plane ⟹ l ∥ Normal

⟹ $\vec{\boldsymbol{n}} = k\,\vec{\boldsymbol{u}}$

Direction of normal is same as direction of line.

10. To find the angle ‘ θ ‘ between line ‘ l ‘ (AQ) and plane ‘ P ‘.

$\vec{n}$ is normal to the Plane.

line l : $\vec{r} = \vec{a} + \lambda \vec{u}$ ----------- (i)

Plane P : $\vec{r} \cdot \vec{n} = d$ -------------- (ii)

Now $\angle CAQ = \frac{\pi}{2} - \theta$

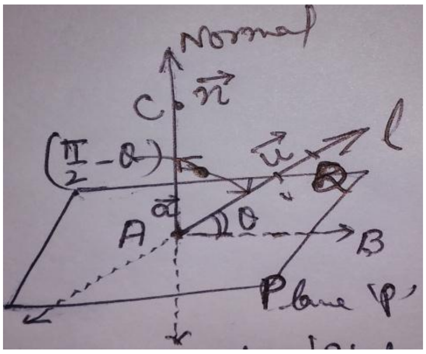
$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{n} \cdot \vec{u}}{|\vec{n}|\,|\vec{u}|} = k$ (let)

Or $\sin\theta = k$

$\theta = \sin^{-1}(k)$

Case I : Let line ‘ l ‘ intersects plane ‘P’ at a point A.

And Let the plane containing the line ‘ l ‘ and normal $\vec{n}$ intersects the plane ‘ P ‘ in the line $\overleftrightarrow{AB}$

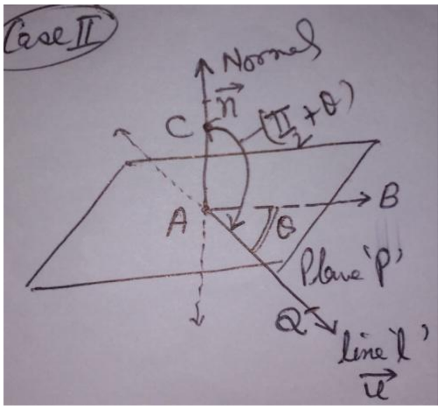


Then the required angle ‘ θ ‘ is between ‘ l ‘ and line AB.

$\theta = \angle QAB$

Hence , the angle between Normal and line ‘ l ‘ = $\left(\frac{\pi}{2} - \theta\right)$

Case II : If the angle between Normal and the line is obtuse.



Take $\angle CAQ = \left(\frac{\pi}{2} + \theta\right)$

$\cos\left(\frac{\pi}{2} + \theta\right) = \frac{\vec{n} \cdot \vec{u}}{|\vec{n}|\,|\vec{u}|} = -k$ (let)

$\Rightarrow -\sin\theta = -k$

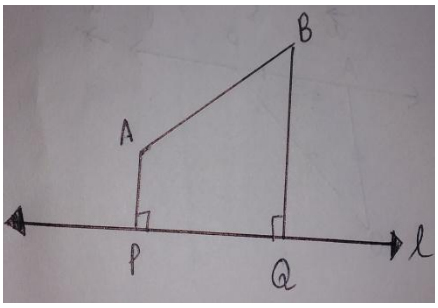
$\Rightarrow \sin\theta = k$

$\theta = \sin^{-1}(k)$

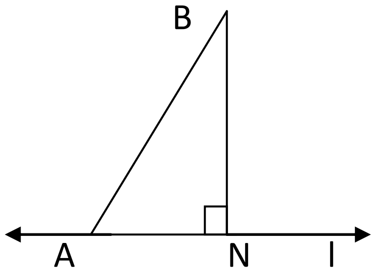
SOME IMPORTANT CONCEPTS

1. Projection of a segment of a line :

Projection of $\overline{AB}$ on l = PQ



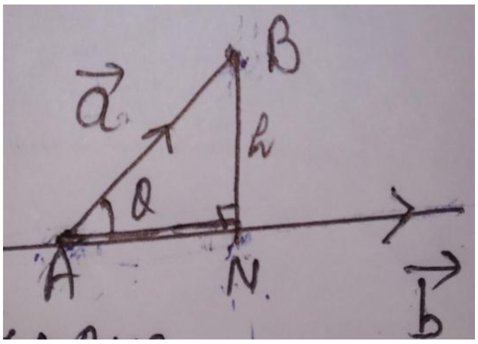
2. Projection of $\overline{AB}$ on line l = AN



3. Let $\overrightarrow{AB} = \vec{a}$

AN = Projection of AB on $\vec{b}$

$\therefore \quad AN = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$



In right $\Delta \ ANB$

$\because \quad \frac{AN}{|\vec{a}|} = \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \ |\vec{b}|}$

$\Rrightarrow$ AN $= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

<u>To Solve two equation in three variables</u> (CROSS MULTIPLICATION METHOD)

$$\mathbf{a_1 x + b_1 y + c_1 z = 0}$$
$$\mathbf{a_2 x + b_2 y + c_2 z = 0}$$

Note : $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{matrix} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{matrix}$$

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{z}{a_1 b_2 - a_2 b_1}$$

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{z}{a_1 b_2 - a_2 b_1} = \lambda$$

<u>ALTERNATE METHOD</u>

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{-(a_1 c_2 - a_2 c_1)} = \frac{z}{a_1 b_2 - a_2 b_1}$$

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{-(a_1 c_2 - a_2 c_1)} = \frac{z}{a_1 b_2 - a_2 b_1} = \lambda$$

APPLICATION :

To Solve for a, b , c

Q6. a + 2 b + 3 c = 0
2 a + b - 2 c = 0

$$\frac{a}{2\,(-2)-1 \times 3} = \frac{b}{3 \times 2-(1)(-2)} = \frac{c}{1 \times 1-2 \times 2}$$

Or $\frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = k$

a = -7 k

b = 8 k or $k \begin{pmatrix} -7 \\ 8 \\ -3 \end{pmatrix}$

c = -3 k

Q9. Solve :

3a + b - c = 0
-a + 2b - c = 0

$$\frac{a}{1(-1)-2\,(-1)} = \frac{b}{(-1)(-1)-(-1) \times 3} = \frac{c}{3 \times 2-(-1) \times 1}$$

$$\frac{a}{-1+2} = \frac{b}{1+3} = \frac{c}{6+1}$$

a: b : c = 1 : 4 : 7

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$