Probability and Statistics 1

Discrete Random Variable

Binomial and Geometric Prob.

Revision

Sp-20 | M-20 | S-20 | M-19 | S-19 | W-19

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Example 1: A book club sends 6 paperback and 2 hardback books to Mrs. Hunt. She chooses 4 of these books at random to take with her on holiday. The random variable $X$ represents the number of paperback books she chooses.

(a) Show that the probability that she chooses exactly 2 paperback books is $\frac{3}{14}$. -- (2)
(b) Draw up the probability distribution table for $X$. -- (3)
(c) You are given that $E(X) = 3$; Find $Var(X)$. -- (2)

Solution (a) Paperback books = 6, Hardcover books = 2; Total 8
No of books chosen = 4

$P(\text{Exactly 2 paperback}) = \frac{\binom{6}{2} \times \binom{2}{2}}{\binom{8}{4}} = \frac{15}{70} = \frac{3}{14}$

(b) | $X$ | 2 | 3 | 4 |
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Prob($X$)</td>
<td>$\frac{6\times2}{8\times7} = \frac{3}{14}$</td>
<td>$\frac{6\times3}{8\times7} = \frac{9}{14}$</td>
<td>$\frac{6\times4}{8\times7} = \frac{3}{14}$</td>
</tr>
</tbody>
</table>

Total 4 books chosen = 4
No of hardbacks = 2
Minimum paperback = 2

(c) $Var(X) = E[X^2] - (E[X])^2$

$= 2 \times\frac{2^2}{14} + 3 \times\frac{3^2}{14} + 4 \times\frac{4^2}{14} - 3^2$

$= \frac{12}{14} + \frac{72}{14} - 9 = \frac{3}{2}$ or $0.429\checkmark$

Example 2: A fair six-sided die, with faces marked 1, 2, 3, 4, 5, 6, is thrown repeatedly until a 3 is obtained.

Find the probability that obtaining a 3 requires fewer than 7 throws, -- (2)

Solution: $P(\text{getting 3 in each throw}) = p = \frac{1}{6}$

To find $P(X < 7) = (1 - \frac{1}{6})^6 = 0.665\checkmark$ [Given $E(X) = 3$]

$P(X < 7) = P(X \leq 6) = p + q \times p + q \times q \times p + q \times q \times q \times p + q \times q \times q \times q \times p$

$= p \left[ 1 + q + q^2 + \ldots + q^6 \right] = p \left[ \frac{1 - (q \times 6)}{1 - q} \right] = \frac{n \times (1 - q \times 6)}{q}$

- "Geometric distribution" $= p \left[ -\frac{1}{1-0.6} \right] = (1 - 0.6)^{-1} \checkmark$
Example 3: An ordinary fair die is thrown repeatedly until 1 or 6 is obtained.

(a) Find the prob. that it takes at least 3 throws but no more than 5 throws to obtain a 1 or 6.

On another occasion die is thrown 3times, the random variable X is the number of times that a 1 or a 6 is obtained.

(b) Draw up the prob. distribution table for X.

(c) $E(X) = 3\frac{1}{3}$

Solution: Sample space $S = \{1, 2, 3, 4, 5, 6\}$

In a single throw $p = P(1 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$

and $q = 1 - \frac{1}{3} = \frac{2}{3}$

(a) at least 3 but not more than 5 throws to get 1 or 6

$$P(3 \leq X \leq 5) = q^5 + q^3 + q^5$$

Alternate method:

Using Geometric distribution,

$$P(X \leq x) = 1 - q^x$$

$$= q^5(1 + q + q^2) = \left(\frac{2}{3}\right)^{\frac{1}{3}} (1 + \frac{2}{3} + \frac{2}{3})$$

$$= \frac{6}{27} \times \frac{9}{9} = \frac{72}{243}$$

(b) Using Binomial distribution:

$n = 3$, $p = \frac{1}{3}$, $q = \frac{2}{3}$

as the prob. of getting 1 or 6 is same in each throw.

$$= \left(\frac{2}{3}\right)^2 \left(1 - \left(\frac{2}{3}\right)^3\right) = \frac{4}{9} \left(1 - \frac{8}{27}\right)$$

$$= \frac{12}{27} = \frac{4}{9} \times \frac{9}{27}$$

$$= \frac{72}{243}$$

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>$n_0 \cdot p^0 q^3$</td>
<td>$\binom{3}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$</td>
<td>$\binom{3}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$</td>
<td>$\binom{3}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0$</td>
</tr>
<tr>
<td></td>
<td>$3c_0 \left(\frac{2}{3}\right)^3$</td>
<td>$\frac{5}{27}$</td>
<td>$\frac{6}{27}$</td>
<td>$\frac{1}{27}$</td>
</tr>
</tbody>
</table>

$E(X) = \sum p_i \cdot x_i = 0 \cdot \frac{5}{27} + 1 \cdot \frac{5}{27} + 2 \cdot \frac{6}{27} + 3 \cdot \frac{1}{27} = \frac{27}{27} = 1$

Alternate method:

for Binomial distribution $E(X) = np = 3 \times \frac{1}{3} = 1$
Example 4: In Greenon, 70% of the adults own a car. A random sample of 8 adults from Greenon is chosen. Find the probability that the number of adults in this sample who own a car is less than 6. 

\[ P(M - 20/52 | Q5(a)) \]  

Solution: \( P(\text{adult has a car}) = 70\% \Rightarrow p = 0.7 \Rightarrow q = 1 - 0.7 = 0.3 \); \( n = 8 \)

\[ P(x < 6) = 1 - P(x \geq 6) = 1 - \sum_{k=6}^{8} \binom{8}{k} (0.7)^k (0.3)^{8-k} \]

\[ = 1 - 0.55177 = 0.448 \]

(using Binomial Prob.)

\[ P(M) = \binom{8}{x} p^x q^{8-x} \]

Example 5: A fair three-sided spinner has sides numbered 1, 2, 3. A fair five-sided spinner has sides numbers 1, 2, 2, 3, 3. Both spinners are spun once. For each spinner, the number on the side on which it lands is noted. The random variable \( X \) is the larger of the two numbers if they are different, and their common value if they are the same.

(a) Show that \( P(X = 3) = \frac{4}{15} \)  

(b) Draw up the prob. distribution table for \( X \)  

(c) Find \( E(X) \) and \( Var(X) \)

\[ E(X) = \frac{52}{15} \quad Var(X) = \frac{4}{45} \]

\( (1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 2), (3, 1) \)

\( n = \text{Total number of outcomes} = 3 \times 5 = 15 \)

\( X = 3 \) has come \( 4 \) times.

\( P(X = 3) = \frac{4}{15} \)

\( X \)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>6/15</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>6/15</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>4/15</td>
</tr>
</tbody>
</table>

\[ Var(X) = \frac{22}{45} \quad (or \quad 0.489) \]

(c) \( E(X) = \sum \frac{x}{n} \)

\[ = 1 \cdot \frac{3}{15} + 2 \cdot \frac{6}{15} + 3 \cdot \frac{5}{15} \]

\[ = \frac{6}{15} + \frac{12}{15} + \frac{15}{15} \]

\[ = \frac{23}{15} \]

\[ Var(X) = \sum (x - E(X))^2 \]

\[ = \frac{2}{15} \cdot \frac{12}{15} + \frac{6}{15} \cdot \frac{9}{15} + \frac{5}{15} \cdot \frac{8}{15} \]

\[ = \frac{84}{15} = \frac{28}{5} \]

\[ Var(X) = \frac{22}{45} \]
Example 6: On any given day, the prob. that Moena messages her friendasha is 0.72.

(a) Find the prob. that for a random sample of 12 days Moena messagesasha on no more than 9 days. --[3]
(b) Moena messagesasha on 1 January. Find the prob. that the next day on which she messagesasha is 5 January. --[1]

\[ P = 0.72 \Rightarrow q = 1 - 0.72 = 0.28, n = 12. \] (Using Binomial probability)

\[ P(X \leq 9) = 1 - P(10, 11, 12) \]
\[ = 1 - \binom{12}{10} (0.72)^{10} (0.28)^2 + \binom{12}{11} (0.72)^{11} (0.28) + 0.72^{12} \]
\[ = 0.96 \times \sqrt{ } \]

(b) Jan 1 \[ \quad \] Jan 2 \[ \quad \] Jan 3 \[ \quad \] Jan 4 \[ \quad \] Jan 5 \[ \quad \] Jan 6 \[ \quad \] Jan 7 \[ \quad \] Jan 8 \[ \quad \] Jan 9 \[ \quad \] Jan 10 \[ \quad \] Jan 11 \[ \quad \] Jan 12 \[ \quad \]
\[ P(\text{message on 5 Jan}) = q^4 p = (0.28)^4 \times 0.72 = 0.0158 \times \sqrt{ } \]

Example 7: The score when two fair six-sided dice are thrown is the sum of
the two numbers on the upper faces.

(a) Show that the prob. that the score is 4 is \[ \frac{1}{12} \]. --[2]

The two dice are thrown repeatedly until a score of 4 is obtained.

(b) Find mean of \( X \) --[1]

(c) Find the prob. that a score of 4 is first obtained on the \( 6 \)th throw. --[1]

(d) Find \( P(X < 8) \)

\[ n = 36 \]
\[ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}, \frac{21}{2}, \frac{23}{2}, \frac{25}{2}, \frac{27}{2}, \frac{29}{2}, \frac{31}{2}, \frac{33}{2}, \frac{35}{2} \]
\[ P(\text{score} 4) = \frac{3}{36} = \frac{1}{12} = P \]

(b) Geometric Postb, \( p = \frac{1}{2} \)

Mean of \( X = \frac{1}{p} = \frac{1}{\frac{1}{12}} = 12 \)

(Using Geometric Prob.)

(c) \( P(4 \text{ on } 6 \text{th throw}) = \frac{6}{36} \times p^6 \times p = \frac{7}{12} \)
Example 8. A company produces small boxes of sweets that contain 5 jelly beans and 3 chocolate. Jameel chooses 3 sweets at random from a box.

(a) Draw up the probability distribution table for the number of jelly beans that Jameel chooses.

The company also produces large boxes of sweets. For any large box, the prob that it contains more jelly beans than chocolate, is 0.64. 10 large boxes are chosen at random.

(b) Find the prob that no more than 7 of these boxes contain more jelly beans than chocolate.

Solution: No. of jelly beans = 5
No. of chocolate = 3
3 sweets are chosen. \( \rightarrow \) Combination \( \binom{n}{k} \)

<table>
<thead>
<tr>
<th>No. of Jellybeans</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X) )</td>
<td>( \frac{5!}{5!} )</td>
<td>( \frac{5!}{1!3!} )</td>
<td>( \frac{5!}{2!2!} )</td>
<td>( \frac{5!}{3!} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{56} )</td>
<td>( \frac{15}{56} )</td>
<td>( \frac{30}{56} )</td>
<td>( \frac{10}{56} )</td>
</tr>
</tbody>
</table>

\( \frac{1}{56} \times 8 \times 7 \times 6 \times 3 \times 2 \times 1 = 56 \)

(b) \( p = 0.64, \quad q = 1 - 0.64 = 0.36, \quad n = 10 \)

\[ P(X \leq 7) = 1 - P(8, 9, 10) = 1 - \left( \frac{10}{C_8} \left(0.64\right)^8 \left(0.36\right)^2 + \frac{10}{C_9} \left(0.64\right)^9 \left(0.32\right)^1 + 0.64^{10} \right) \]

Using Binomial Prob.

\[ P(X = x) = \binom{n}{x} p^x, \quad q = n - x \]

Example 9: In a certain large college, 33% of students own a car.

(a) 3 students from the college are chosen at random. Find the prob. that all three students own a case. \( \rightarrow \)

(b) 16 students from the college are chosen at random. Find the prob. that the number of those students who own a car is at least 3 and at most 4.

Solution:

Using Binomial Prob. \( P(X = x) = \binom{n}{x} p^x, \quad q = n - x \)

(a) \( n = 3, \quad p = 0.22, \quad q = 0.78 \)

\[ P(X = 3) = \binom{3}{3} p^3 = (0.22)^3 \]

(b) \( n = 16, \quad p = 0.22, \quad q = 0.78 \)

\[ P(3 \leq x \leq 4) = \binom{16}{3} (0.22)^3 (0.78)^{13} + \binom{16}{4} (0.22)^4 (0.78)^{12} \]

\[ = 0.63 \]

Scanned with CamScanner
Example 10. A fair four-sided spinner has edges numbered 1, 2, 3, 4. A fair three-sided spinner has edges numbered -2, -1, 1. Each spinner is spun and the number on the edge on which it comes to rest is noted. The random variable $X$ is the sum of the two numbers that have been noted.

(a) Draw the prob. distribution table for $X$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{2}{12}$</td>
<td>$\frac{2}{12}$</td>
<td>$\frac{3}{12}$</td>
</tr>
</tbody>
</table>

(b) Find $\text{Var}(X)$.

\[ \text{Var}(X) = \sum (x_i - \mu)^2 \cdot P(X) \]
\[ = \frac{1}{12} (1 + 0 + 3 + 4 + 6 + 4) - \left( \frac{4}{3} \right)^2 = \frac{9}{18} = 0.67 \]

Example 11. A pair of fair coins is thrown repeatedly until a pair of tails is obtained. The random variable $X$ denotes the number of throws required to obtain a pair of tails.

(a) Find the expected value $\mu$.

\[ \mu = \frac{1}{12} (1 + 2 + 3 + 4) = \frac{10}{12} = 0.67 \]

(b) Find the prob. that exactly 3 throws are required to obtain a pair of tails.

\[ P(X = 3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \]

(c) Find the prob. that fewer than 6 throws are required to obtain a pair of tails.

\[ P(X < 6) = 1 - 0.5^6 = 0.9763 \]

Solution: A pair of coins is thrown -

\[ \text{S} = \{HH, HT, TH, TT\} \]

\[ p = P(\text{both TT}) = \frac{1}{4}, \quad P = \frac{3}{4}, \quad q = \frac{1}{2} \]

\[ E(X) = \frac{1}{p} = \frac{1}{\frac{1}{4}} = 4 \]

* (Geometric Probability) \[ ; p (x < 5) = 1 - 0.5^5 \]
Example 12. The random variable \( X \) takes the values \(-1, 1, 2, 3\) only. The prob. that \( X \) takes the value \( x \) is \( kx^2 \), where \( k \) is a constant.

i) Draw up the prob. distribution table for \( X \), in terms of \( k \), and find the value of \( k \). 

\[ \begin{array}{c|cccc} \hline x & -1 & 1 & 2 & 3 \\ \hline p(x) = p & k & k & 4k & 9k \\ \hline \end{array} \]

Now \( \sum p_i = 1 \Rightarrow 15k = 1 \Rightarrow k = \frac{1}{15} \checkmark \)

ii) Find \( E(X) \) and \( \text{Var}(X) \)

\[ E(X) = \sum p_i x_i = -1 \cdot k + 1 \cdot k + 2 \cdot 4k + 3 \cdot 9k \]
\[ = 35k = 35 \cdot \frac{1}{15} = \frac{7}{3} \checkmark \]

\[ \text{Var}(X) = \sum (p_i x_i^2 - (E(X))^2) = 99k - \frac{9}{9} = 99 \cdot \frac{1}{15} - \frac{9}{45} = \frac{54}{45} = 1.16 \checkmark \]

Example 13. The results of a survey by a large supermarket show that 35% of its customers shop online.

i) Six customers are chosen at random. Find the prob. that more than three of them shop online. 

\[ P(X > 3) = P(X = 4, 5, 6) = 6 \cdot (0.35)^4 \cdot (0.65)^2 + \binom{6}{5} (0.35)^5 \cdot (0.65) + (0.35)^6 \]
\[ = 0.11 \checkmark \]

ii) For a random sample of \( n \) customers, the prob. that at least one of them shops online is greater than 0.95. Find the least possible value of \( n \).

\[ P(\text{at least one}) = 1 - P(0) \]

\[ 0.95 = 1 - (0.65)^n \]

\[ \Rightarrow n \geq \log 0.05 = 6.95 \]

\[ \Rightarrow n = 7 \checkmark \]

Note: \( \log 0.05 < 0 \), when we divide by a negative no, the sign of inequality reverses.
Example 14: In a certain country, the prob. that a child owns a bicycle is 0.65. A random sample of 15 children from this country is chosen. Find the prob. that more than 12 own a bicycle.

\[ P(X = n) = \binom{15}{x} (0.65)^x (0.35)^{15-x} \]

\[ \text{Solution: } P(X > 12) = P(X = 13, 14, 15) = \binom{15}{13} (0.65)^{13} (0.35)^2 + \binom{15}{14} (0.65)^{14} (0.35) + (0.65)^{15} \]

\[ = 0.0617 \]

Example 15: At a fun fair, Amy pays $1 for two attempts to make a bell ring by shooting at it with a water pistol.
- If she makes the bell ring on her first attempt, she receives $3 and stops playing. This means that overall she has gained $2.
- If she makes the bell ring on her second attempt, she receives $1.50 and stops playing. This means that overall she has gained $0.50.
- If she does not make the bell ring in two attempts, she has lost her $1.

The prob. that Amy makes the bell ring on any attempt is 0.2, independently of others.

(i) Show that the prob. that Amy loses her original $1 is 0.64.

(ii) Complete the prob. distribution table for the amount that Amy gains.

<table>
<thead>
<tr>
<th>Amy's gain ($)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.64</td>
<td></td>
</tr>
</tbody>
</table>

(iii) Calculate Amy's expected gain.

\[ E(X) = -1 \times 0.64 + 0.5 \times 0.16 + 2 \times 0.2 = -0.64 + 0.08 + 0.4 = -0.64 + 0.48 = -0.16 \]

\[ \text{loss of 16 cents} \]
Example 16: The probability that Janice will buy an item online in any week is 0.35. Janice does not buy more than one item online in any week.

(i) Find the prob. that, in a 10-week period, Janice buys at most 7 items.

(ii) The prob. that Janice buys at least one item online in a period of n weeks is greater than 0.99. Find the smallest possible value of n.

\[ n > \frac{-4.605}{0.1230} = 101.7 \approx 102 \]

Solution: \( p = 0.35, q = 0.65, n = 10 \)

(i) \[ P(2 \leq 7) = 1 - P(8,9,10) \]

(ii) \[ P(\text{at least one}) = 1 - (0.65)^n > 0.99 \] Given

\[ n > 0.01 > (0.65)^n \]

\[ \ln(0.01) > n \ln(0.65) \]

\[ n > \frac{-0.430}{-0.430} = 1 \]

\[ n > 1 \approx 1 \]

Example 17: On average, 34% of the people who go to a particular theatre are men. A random sample of 14 people who go to theatre is chosen. Find the prob. that at most 2 people are men.

Solution: \( n = 14, p = 0.34, q = 0.66, \text{ use Binomial Prob.} \)

\[ P(1 \leq 2) = P(0,1,2) = (0.66)^{14} + 14(0.34)(0.66)^{13} + 14(0.34)^2(0.66)^{12} \]

\[ = 0.0963 \checkmark \]

\[ n \ln(0.65) < \ln(0.01) \]

\[ -0.430 > n < -4.605 \]

\[ n > \frac{-4.605}{-0.430} \]

\[ (\text{note: divide by -0.430, a negative, the sign of inequality changes.}) \]

\[ n > 10.7 \approx 11 \]
Example 18: A fair five-sided spinner has sides numbered 1, 1, 1, 2, 3.
A fair three-sided has sides numbered 1, 2, 3. Both spinners are
shaken once and the score is the product of the numbers on the sides
the spinner lands on.

(i) Draw up the probability distribution table for the score.
(ii) Find the mean and the variance of the score.
(iii) Find the probability that the score is greater than the mean score.

Solution:

<table>
<thead>
<tr>
<th>Score</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 1, 2, 3</td>
<td>[ \frac{3}{15}, \frac{4}{15}, \frac{3}{15}, \frac{4}{15}, \frac{1}{15} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>[ \frac{3}{15}, \frac{4}{15}, \frac{3}{15}, \frac{4}{15}, \frac{1}{15} ]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Total outcomes: 5

Mean = \[ \frac{1}{5} \sum Xi \] = \[ \frac{1}{5} \times 1 + 1 + 1 + 2 + 3 = \frac{7}{5} \]

Variance = \[ \frac{1}{5} \sum (Xi - \mu)^2 \] = \[ \frac{1}{5} \left( (1 - \frac{7}{5})^2 + (1 - \frac{7}{5})^2 + (1 - \frac{7}{5})^2 + (2 - \frac{7}{5})^2 + (3 - \frac{7}{5})^2 \right) = \frac{16}{25} \]

\( P(X > \mu) = P(4, 6, 9) = \frac{3}{15} + \frac{4}{15} + \frac{1}{15} = \frac{2}{5} = 0.4 \)

Example 19: Arman has designed a new logo for a sports wear company. A survey
of a large number of customers found that 42% of customers rated the
logo as good.

(i) A random sample of 10 customers is chosen. Find the probability that fewer
than 8 of them rate the logo as good.
(ii) On another occasion, a random sample of n customers of the company
is chosen. Find the smallest value of n for which the probability that at least
one person rates the logo as good is greater than 0.995.

Solution:

\( P(X > 8) = 0.58 \), \( n = 10 \)

\( P(X < 8) = 1 - P(X \geq 8) = 1 - (0.58)^{10} = 0.983 \)

\( \sqrt{ \text{Minimum value of } n \text{ such that } P(X \geq 1) > 0.995 } \)

\[ \log_{0.58} 0.005 = \frac{\log_{10} 0.005}{\log_{10} 0.58} = \frac{-3.3010}{-0.58} = 5.72 \]

\[ n = 10 \]

Note: \[ \log_{0.58} 0.005 < 0 \] when we consider the sign of inequality changes.
Example 20: In a prob. distribution the random variable $X$ takes the values $-1, 0, 1, 2, 4$. The prob. distribution table for $X$ is as follows:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{4}$</td>
<td>$p$</td>
<td>$p$</td>
<td>$\frac{3}{8}$</td>
<td>$4p$</td>
</tr>
</tbody>
</table>

(i) Find the value of $p$.

(ii) Find $E(X)$ and $\text{Var}(X)$.

(iii) Given that $X$ is greater than zero, find the prob. that $X$ is equal to 2.

Solution: $\Sigma p_i = \frac{1}{4} + p + p + \frac{3}{8} + 4p = 1$ (i)

$\Rightarrow 6p + \frac{3}{8} = 1 \Rightarrow p = \frac{1}{16}$

$E(X) = \Sigma x_i P(x_i) = -1 \cdot \frac{1}{4} - \frac{1}{16} + 1 \cdot \frac{1}{16} + 2 \cdot \frac{3}{8} + 4 \cdot 4p = 2.5p$

$\text{Var}(X) = \Sigma (x_i - E(X))^2 P(x_i) = (0 - 2.5)^2 \cdot \frac{1}{4} + (\frac{1}{16} - 2.5)^2 \cdot \frac{1}{16} + (\frac{3}{8} - 2.5)^2 \cdot \frac{3}{8} + (4 - 2.5)^2 \cdot 4p$

Example 21: A fair red spinner has four sides, numbered $1, 3, 3, 3$. A fair blue spinner has three sides, numbered $-1, 0, 2$. When a spinner is spun, the score is the number on the side on which it lands. The spinners are spun at the same time. Let the random variable $X$ denote the score on the red spinner minus the score on the blue spinner.

(i) Draw up the prob. distribution table for $X$.

(ii) Find $\text{Var}(X)$.

Solution:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

$E(X) = \Sigma x_i P(x_i) = -1 \cdot \frac{1}{12} + 0 \cdot \frac{1}{12} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{12} = \frac{3}{4}$

$\text{Var}(X) = \Sigma (x_i - E(X))^2 P(x_i) = (0 - \frac{3}{4})^2 \cdot \frac{1}{12} + (1 - \frac{3}{4})^2 \cdot \frac{1}{4} + (2 - \frac{3}{4})^2 \cdot \frac{1}{12} = \frac{23}{12}$
Example 23. A box contains 3 red balls and 5 white balls. One ball is chosen at random from the box and is not returned to the box. A second ball is now chosen at random from the box.

(i) Find the prob. that both the balls chosen are red. \(-0.17\)

(ii) Show that the prob. that the balls chosen are of different colours is \(15/28\). \(-1.21\)

(iii) Given that the second ball chosen is red, find the prob. that the first ball chosen is red.

The random variable \(X\) denotes the number of red balls chosen.

(iv) Draw up the prob. distribution table for \(X\). \(-1.21\)

(v) Find \(\text{Var}(X)\). \([-1.31]\)

Solution:

Red balls = 3, White balls = 5, Total no. of balls = 8

(i) \(P(\text{RR}) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}\)

(ii) \(P(\text{RW}) + P(\text{WR}) = \frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7} = \frac{15}{28}\)

(iii) \(P(\text{first red/second red}) = \frac{P(\text{RR})}{P(\text{RR}) + P(\text{WR})} = \frac{\frac{3}{8} \times \frac{2}{7}}{\frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7}} = \frac{\frac{3}{8} \times \frac{2}{7}}{\frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7}} = \frac{\frac{3}{28}}{\frac{15}{28}} = \frac{1}{5}\)

(iv) \(E(X) = \sum p_i x_i = 0 \times \frac{1}{28} + \frac{3}{28} \times \frac{3}{28} = \frac{9}{28} - \frac{3}{4}\)

\(\text{Var}(X) = \sum p_i x_i^2 - (E(X))^2 = 0 + \frac{15}{28} + \frac{15}{28} - \left(\frac{3}{4}\right)^2 = \frac{36}{28} - \frac{9}{16} = \frac{45}{112} \approx 0.402\)