S-1

Probability and Statistics

Permutations and Combinations

Revision

SP-20/M-20/S-20/M-19/S-19/W-19

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Example 1: A group of 8 friends travels to the airport in two taxis, P and Q.

Each taxi can take 4 passengers.

(a) The 8 friends divide themselves into two groups of 4, one group for Taxi P and one group for Taxi Q, with Jon and Sarah travelling in the same taxi.

Find the number of different ways in which this can be done. -- 137

<table>
<thead>
<tr>
<th>Taxi P</th>
<th>Taxi Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Back</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>Front</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each taxi can take 1 passenger in front and 3 passengers in the back. Martha sits in front of Taxi P and Jon and Sarah sit in the back of Taxi P next to each other.

(b) Find the number of different seating arrangements that are now possible for the 8 friends. [SP-20/05/86] -- 41

Solution: Jon and Sarah both may sit in Taxi P or Q = 2 ways -- (1)

(a) Now out of remaining 6, two may sit in Jon & Sarah Taxi = 6 ways

Rest of the 4 may sit in another taxi any way = 4C2

\[ \text{Total number of ways} = \binom{6}{2} \times 4 \]

\[ = 2 \times 6 \times 5 \times 1 = 30 \text{ ways} \]

(b) In Taxi P ⫷

Martha sits in front. → 1 way

Sarah & Jon sit at back next to each other

\[ \frac{5!}{2!} \rightarrow \text{may sit in 2 ways} \]

\[ \text{but they can rearrange themselves} \]

\[ = 2 \text{ ways} \]

and the remaining seat will be taken by one of remaining 5 = 5P1 way

\( \text{For Taxi P} = 1 \times (2 \times 2) \times 5 = 20 \text{ ways} \)

The remaining 4 may sit in any way in Taxi Q → 4P4 = 4! = 24 ways

∴ Total number of seating arrangement = 20 \times 24

\[ = 480 \]

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Example 2: The 40 members of a club include Ranuf and Seed. All 40 members will travel to a concert. 35 members will travel in a coach and the other 5 will travel in a car. Ranuf will be in coach, and Seed will be the car.

In how many ways can the members who will in the coach be chosen.

\[ \text{Answer: } \binom{40}{35} = 5.052.252 \]

Solution: Ranuf in coach \( \rightarrow \) remain seat in coach \( = 35-1 = 34 \)

Ranuf in coach & Seed have taken their seat \( \rightarrow \) remaining \( = 40-2 = 38 \).

\[ \text{No of way the member may be in coach } = \binom{38}{34} = \frac{38!}{34!} \frac{4!}{2!} \]

\[ = \frac{38 \times 37 \times 36 \times 35}{4 \times 3 \times 2 \times 1} = 73,815 \]

Example 3: Richard has 3 blue candles, 2 red candles and 6 green candles.

The candles are identical apart from their colours. He arranges 11 candles in a line.

(a) Find the number of different arrangements of the 11 candles if there is a red candle at each end.

(b) Find the number of different arrangements of the 11 candles if all the blue candles are together and red candles are not to together.

Solution:

(a) \( R, \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times R \)

Two red candles are at the end. Remaining 3 blue & 6 green may be arranged any way (but 3 blue are of one kind and 6 green are of another one kind)

\[ \text{Total number of arrangement } = \frac{9!}{3!6!} = 84 \]

(b) Case I when 3 blue one together \( \rightarrow \) 1 bundle of 3 blue & 6 green & 2 red any way

\[ N_0 = \frac{9!}{2!6!} \]

Case II when 3 blue together & 2 red together = \( \frac{8!}{6!} \)

\[ \text{Red number } = \frac{9!}{2!6!} - \frac{8!}{6!} = 352 - 56 = 296 \]
Example 4: Find the number of different arrangements that can be made from the 9 letters of the word JEWELLERY in which
(a) the three Es are together and 2 Ls are together. -- [2]
(b) Find the different arrangements that can be made from the 9 letters of the word JEWELLERY in which the two Ls are not next to each other. [5-20] [5] [2] -- [4]

Solution:

(a) Three Es together & 2 Ls together.
   Effective no → 6
   No. of arrangements = 6! = 720

(b) Total number any way = 9!
   when two Ls together = 8!
   Reg. no. when two Ls not together = \( \frac{9! - 8!}{3!} \)
   \[ = \frac{30240 - 6720}{6} = 3360 \]

Example 5. In a music competition, there are 8 pianists, 4 guitarists, and 6 violinists. 7 of these musicians will be selected to go through to the final. How many different selections of 7 finalists can be made if there must be at least 2 pianists, at least 1 guitarist and more violinists than guitarists. [5-20] [5] [8] [4] -- [4]

Solution: \( \frac{8}{2}, \frac{4}{1}, \frac{6}{2} \) \( \Rightarrow \) total = 18, to be selected = 7, \( 'V > G' \)

Total = \( \frac{8}{2}, \frac{4}{1}, \frac{6}{2} \)

- \( \Rightarrow \) \( \frac{2}{2}, \frac{1}{1}, \frac{4}{1} \)
  \( = \frac{8}{2} \times \frac{4}{1} \times \frac{6}{2} = 8 \times 4 \times 3 = 680 \)

- \( \Rightarrow \) \( \frac{1}{2}, \frac{2}{1}, \frac{3}{1} \)
  \( = \frac{8}{2} \times \frac{4}{1} \times \frac{6}{3} = 8 \times 4 \times 2 = 80 \)

- \( \Rightarrow \) \( \frac{1}{2}, \frac{1}{1}, \frac{3}{1} \)
  \( = \frac{8}{2} \times \frac{4}{1} \times \frac{6}{3} = 8 \times 4 \times 2 = 80 \)

- \( \Rightarrow \) \( \frac{1}{2}, \frac{1}{1}, \frac{2}{1} \)
  \( = \frac{8}{2} \times \frac{4}{1} \times \frac{6}{2} = 8 \times 4 \times 3 = 90 \times 15 = 1350 \)
Example 6
(a) Find the number of different ways in which the 10 letters of the word SUMMERTIME can be arranged so that there is an E at the beginning and an E at the end.

(b) Find the number of different ways in which the 10 letters of the word SUMMERTIME can be arranged so that E's are not together.

Solution: SUMMERTIME (S = 1)

Total = 10, 2E and 3M

Total number of ways any way = \( \frac{10!}{2! 3!} \) = 6720

(b) Total 10, 2E and 3M

Total number of ways any way = \( \frac{10!}{2! 3!} \) = 6720

and with 2E together = \( \frac{9!}{3!} \) = 604,800

Leave the number of ways when 2E are not together = \( \frac{10!}{2! 3!} - \frac{9!}{3!} \) = 24,920

Example 7
(a) Find the number of different possible arrangements of the 9 letters in the word CELESTIAL.

(b) Find the number of different arrangements of the 9 letters in the word CELESTIAL in which the first letter is E, the fifth letter is T and the last letter is E.

(c) Find the prob. that a randomly chosen arrangement of the 9 letters in the word CELESTIAL does not have the two E's together.

(d) Find the number of different selections if the 5 letters include at least one E and almost one L.

Solution (a) \( \frac{9!}{2!} = 907,200 \) [1, 2L & 2L]

in Total 9.

(b) C ----- T ----- E

\( \frac{6!}{2!} = 360 \) [2L in 6 letters]

(c) 2E together = \( \frac{8!}{2!} = 20,736 \)

2E not together = \( 907,200 - 20,736 \) = 886,464

(d) EL ---- = \( \frac{5}{2} = 10 \)

\( \alpha \) EEEL ---- = \( \frac{5}{2} = 10 \)

EE ---- = \( \frac{5}{2} = 10 \)

1. Total number of ways = \( 10 + 10 + 5 + 10 = 35 \)
Example 8: Find the number of different arrangements that can be made of all 9 letters in the word CAMERAMAN in each of the following cases.
(i) There are no restrictions. -- [5]
(ii) The As occupy the 1st, 5th, and 9th position. -- [13]
(iii) There is exactly one letter between the Ms. -- [14]
(iv) Three letters are selected from the 9 letters of the word CAMERAMAN.
(v) Find the number of different selections if the three letters include exactly one M and exactly one A.

\[ \binom{M-1}{1} \times \binom{A-3}{1} \times \binom{C-1}{3} \]

\[
\frac{9!}{3! \cdot 2! \cdot 1!} = 30,240
\]

Solution:
(i) CAMERAMAN

\[
\frac{9!}{3! \cdot 2! \cdot 1!} = 30,240
\]

(ii) A \times \times \times \times A \times \times A \times \times A

As are fixed put 4 remaining 6

\[
\binom{6!}{4! \cdot 2!} = 360
\]

(iii) M x M x x x x x x x

Two Ms are fixed and in the middle

\[
\binom{7!}{2! \cdot 3!} = 5,880
\]

(iv) MA x Then cut the rest after

\[
\frac{8!}{4!} \times \binom{4!}{3!} = 4,032
\]

(v) M x x Two letters out of four

Case I

\[
\binom{4!}{2! \cdot 2!} = 6
\]

Case II

\[
\binom{4!}{1! \cdot 3!} = 4
\]

Case III

\[
\binom{4!}{1! \cdot 1! \cdot 2!} = 1
\]

Case IV

\[
\binom{4!}{1! \cdot 1! \cdot 1!} = 1
\]

Case V

\[
\binom{4!}{1! \cdot 1! \cdot 1!} = 1
\]

Total number selections

\[
6 + 4 + 1 + 1 + 1 + 1 = 16
\]
Example 9: Freddie has 6 toy cars and 3 toy buses, all different. He chooses 4 toys to take on holiday with him.

(i) In how many different ways can Freddie choose 4 toys? \(\text{--[1]}\)

(ii) How many of these choices will include both his favourite car and his favourite bus? \(\text{--[2]}\)

Freddie arranges these 9 toys in a line.

(iii) Find the number of these arrangements if the buses are all next to each other. \(\text{--[3]}\)

(iv) Find the number of possible arrangements if there is a car at each end of the line and no buses next to each other. \(\text{--[3]}\)

S 19/6 1/08

Solution: 6 toy cars & 3 toy buses, total = 9 all different

(i) Choose 4 toys any way = \(\text{C}_9^4\)

\[= 126\] \(\checkmark\)

(ii) favourite car & bus done already

selected, out of remaining any 2

\[= \text{C}_7^2 = 21\] \(\checkmark\)

(iii) \((B_1, B_2, B_3)\) as if one group.

as if \(N\) ow total \(116 = 7\)

but bus: multiply can be arranged

\[= \frac{7!}{4!} \times 3! = 30240\] \(\checkmark\)

(iv) \(C_1 - C_2 - C_3 - C_4 - C_5 - C_6\) (56A + 3)

Two cars at the end and four in the middle, all 6 \(\rightarrow 6!\)

as no buses are next to each other, five gaps between 6 cars

to buses may be placed \(\rightarrow 5P_3\)

\[= 6! \times 5P_3 = 43200\] \(\checkmark\)
Example 10(a) A group of 6 teenagers go boating. There are three boats available, one boat has room for 3 people, one has room for two people and one has room for 1 person. Find the number of different ways the group of 6 teenagers can be divided between the three boats.

(b) Find the number of different 7-digit numbers which can be formed from the seven digits 2, 2, 3, 3, 7, 7, 8 in each of the following cases,

(i) The odd digits are together and even digits are together.

(ii) The 2’s are not together.

[3-19/62/2/17 -- 14]

Solution (a) $6 \times 3 \times 2 \times 1 = 20 \times 3! = 60$  

(b)(ii) No. of odd = 2  
No. of even = 4  

$\frac{2!}{2!} \times \frac{4!}{2!} \times 2$ (interchanged) 

Total no. of arrangements $= \frac{7!}{2!} = 4,200$  

(i) No. with 2’s together $= \frac{6!}{2!} = 120$  

With 2’s not together $= 4,200 - 120 = 4,080$  

= 300 ways.

Example 11: Mr. and Mrs. Kennedy and their 5 children all go to watch a football match, together with their friends Mr. and Mrs. Ugama and their 2 children. Find the number of ways in which all 11 people can line up at the entrance in each of the following cases.

(i) Mr. Kennedy stands at one end of the line and Mrs. Ugama stands at the other end.

(ii) The five Kennedy children all stand together and the Ugama children both stand together.

[3-19/63/03 -- 3]

Solution: $K \times \ldots \times \times \times \times \times \times \times \times \times \times \times \times U$

(i) Kennedy and Ugama at the two ends who can interchange — and rest of the 9 in any way

$= 9! \times 2 = 725,760$

(ii) $(K_1 K_2 K_3 K_4 U_1 U_2)$

Can be rearranged among themselves.

$= \frac{6!}{2!} \times 12 = 1,280$  

Reft 4
Example 12(i) Find the number of ways a committee of 6 people can be chosen from 8 men and 4 women if there must be at least twice as many men as there are women on the committee.

(ii) Find the number of ways a committee of 6 people can be chosen from 8 men and 4 women if 2 particular men refuse to be on the committee together.

Solution

<table>
<thead>
<tr>
<th>No. of men in</th>
<th>Ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2 in $C_6^4 \times C_4^2 = 420$ ways</td>
</tr>
<tr>
<td>5 (double)</td>
<td>1 in $C_6^5 \times C_4^2 = 224$ ways</td>
</tr>
<tr>
<td>6</td>
<td>0 in $C_6^6 \times C_4^2 = 28$ ways</td>
</tr>
</tbody>
</table>

Total ways = 420 + 224 + 28 = 672 ways

(iii) Total no. of selections = $C_6^6 \times C_4^2 = 924$ ways

Example 13(i) Find the number of different ways in which all 12 letters of the word STEEPLECHASE can be arranged so that all 4 Es are together.

(ii) Find the number of different ways in which all 12 letters of the word STEEPLECHASE can be arranged so that the Ss are not next to each other.

(iii) Find the number of different selections if the four letters include exactly one S.

Solution

(i) 4 Es, 3 Ss and 5 all different

4 Es together = $C_4^4 \times C_3^3 \times C_6^6 = 91$ ways

Total ways = $91 \times 1 \times 1 \times 720 = 61440$

(ii) Total no. of ways (any way) = $12! = 9979200$

with Ss together = $12! / 2! = 1663200$

with Ss not together = $9979200 - 1663200$

$= 8316000$

(iii) SEE = 1

SEE (out of 4 Es) = $C_4^4 = 6$

SE (out of 4 Es) = $C_3^3 = 15$

S --- = $C_6^3 = 20$

Total no. = 1 + 6 + 15 + 20

$= 42$
Example 14. (i) Find the number of different ways in which the 9 letters of the word TOADSTOOL can be arranged so that all three Os are together and both Ts are together. \( \text{---[1]} \)

(ii) Find the number of different ways in which the 9 letters of the word TOADSTOOL can be arranged so that Ts are not together. \( \text{---[2]} \)

(iii) Find the prob that a randomly chosen arrangement of the 9 letters of the word TOADSTOOL has a T at the beginning or T at the end. \( \text{---[2]} \)

(iv) Five letters are selected from the 9 letters of the word TOADSTOOL, find the number of different selections if the 5 letters include at least 2 Os and at least 1 T. \( \text{[W-19/63]Q7} \text{---[4]} \)

Solution: TOADSTOOL \( \rightarrow 3T, 3O, 4 \) different

(i) \( \binom{3}{2} \binom{5}{3} \binom{6}{2} \binom{6}{2} \binom{6}{2} \binom{6}{2} \text{ as if '6'} \n\text{arrangements} = 6! = 720 \checkmark \)

(ii) Total no of arrangements \( = \frac{9!}{3!3!} = 30240 \checkmark \)

No sol with 2Ts together \( = \frac{8!}{3!} = 6720 \checkmark \)

\( \therefore \text{ways T are not together} = 30240 - 6720 = 23520 \checkmark \)

(iii) \( \binom{3}{0} \binom{6}{3} \binom{7}{2} \text{ (seven)} \)

\( \text{Total no of way any way} = 30240 \text{ from (1)} \)

\( \therefore \text{Prob} = \frac{840}{30240} = \frac{1}{36} \text{ or (0.0278)} \)

(iv) \( \binom{3}{2} = 6 \)
\( \binom{3}{1} = 4 \)
\( \binom{3}{3} = 1 \)

\( \therefore \text{Total number of ways} = 6 + 4 + 4 + 1 = 15 \checkmark \)
Example 15: (i) How many different arrangements are there of the 9 letters in the word CORRIDORS?

(ii) How many different arrangements are there of the 9 letters in the word CORRIDORS in which the first letter is D and the last letter is R or O.

Solution: CORRIDORS → R-3, O-2 and 5 different letters

(i) Letters arrangement = \( \frac{9!}{3!2!} = 30240 \)

(ii) D, O, R

\[ \text{Case I} \rightarrow \frac{2!3!0}{5!} = 1260 \]

\[ \text{Case II} \rightarrow \frac{5!2!0}{3!} = 840 \]

\[ \Rightarrow \text{Total} = 1260 + 840 = 2100 \]

Example 16: A sports team of 7 people is to be chosen from 6 attackers, 5 defenders, and 4 midfielders. The team must include at least 3 attackers, at least 2 defenders and at least 1 midfielder.

(i) How many different ways can the team of 7 people be chosen.

(ii) In how many different ways can the team of 7 be divided into group of 4 and group of 3.

Solution: Total \( \binom{5}{d} \binom{6}{a} \binom{4}{m} \)

(i) Team of 7:

Case I \( A + 2D + 3M \rightarrow \binom{6}{A} \binom{5}{2} \binom{4}{3} = 120 \)

Case II \( A + 2D + 1M \rightarrow \binom{6}{A} \binom{5}{2} \binom{4}{1} = 600 \)

Case III \( A + 3D + 1M \rightarrow \binom{6}{A} \binom{5}{3} \binom{4}{1} = 800 \)

\[ \Rightarrow \text{Total} = 120 + 600 + 800 = 2600 \]

(ii) \( \frac{\binom{4}{A} \cdot 3}{\binom{3}{C_3}} = 35 \)